(1Q1b) Using D'Alembert's principle derived the lagrange's equation of motion for a Conservative force field for a system of N particle.

Solution

let Consider the position of Vector r_i which expressed as the functions of n generalized Cordinates $q_1, q_2, q_3 - - - q_n$ and time (t) as

$$r_i = r_i(q_1, q_2, q_3 - - - q_j, t) \quad (1)$$

$$d\vec{r_i} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r_i}}{\partial q_2} dq_2 + - - - \frac{\partial \vec{r_i}}{\partial q_i} dq_i + \frac{\partial \vec{r_i}}{\partial t} dt.$$

$$d\vec{r_i} = \sum_i \frac{\partial \vec{r_i}}{\partial q_j} dq_j + \frac{\partial \vec{r_i}}{\partial t} dt$$

$$d\vec{r_i} = \sum_j \frac{\partial \vec{r_i}}{\partial q_j} dq_j \quad (2)$$

from The D'Alembert's principle we have -

$$\sum_i (F_i - \dot{p}_i) dr_i = 0 \quad (3)$$

$$\sum_i F_i dr_i - \dot{p}_i dr_i = 0$$

$$\sum_i F_i dr_i = \sum_i F_i \cdot \sum_j dr_i \quad (4)$$

Putting equation (2) into (4)

$$\sum_i F_i dr_i = \sum_i F_i \cdot \sum_j \frac{\partial r_i}{\partial q_j} dq_j$$

$$= \sum_i \sum_j F_i \frac{\partial r_i}{\partial q_j} dq_j$$

But $Q_j = \sum_i F_i \frac{\partial r_i}{\partial q_j}$ — (5) it is called generalized force.

(1)

$$\sum_i F_i dr_i = \sum_j Q_j dq_j \quad \text{--- (6)}$$

Again Consider

$$\sum_i \dot{p} dr_i = \sum_i \dot{p} \cdot \sum_j dr_i \quad \text{--- (7)}$$

Putting equation (2) into (7)

$$\sum_i \dot{p} dr_i = \sum_i \dot{p} \cdot \sum_j \frac{\partial r_i}{\partial q_j} dq_j$$

$$= \sum_i \sum_j \dot{p}_i \frac{\partial r_i}{\partial q_j} dq_j$$

$$\sum_i \dot{p} dr_i = \sum_i \sum_j m_i \ddot{r} \frac{\partial r_i}{\partial q_j} dq_j \quad \text{--- 8}$$

But $\frac{d}{dt}\left[\dot{r}_i \frac{\partial r_i}{\partial q_j}\right] = \ddot{r} \frac{\partial r_i}{\partial q_j} + \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]$

$$\ddot{r} \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}\left[\dot{r}_i \frac{\partial r_i}{\partial q_j}\right] - \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right] \quad \text{--- (9)}$$

Putting equation 9 into (8)

$$\sum_i \dot{p} dr_i = \sum_i \sum_j m_i \left[\frac{d}{dt}\left[\dot{r}_i \frac{\partial r_i}{\partial q_j}\right] - \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]\right] dq_j$$

$$= \sum_i \sum_j \left[\frac{d}{dt}\left[m_i \dot{r}_i \frac{\partial r}{\partial q_j}\right] - m_i \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]\right] dq_j$$

$$= \sum_j \left[\frac{d}{dt}\left[m_i v \frac{\partial v}{\partial q_j}\right] - m_i v \frac{\partial v}{\partial q_j}\right] dq_j$$

(2)

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial q_j}\left(\sum_i \tfrac{1}{2} m_i v^2\right)\right] - \frac{\partial}{\partial q_j}\left(\sum_i \tfrac{1}{2} m v^2\right)\right] dq_j$$

where $T = \sum_i \tfrac{1}{2} m_i v^2$

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial q_j}(T)\right) - \frac{\partial}{\partial q_j}(T)\right] dq_j$$

$$\sum_i \dot{p} \frac{\partial r_i}{\partial q_j} = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] dq_j \quad \text{--- (10)}$$

Substituting 10 & 7 into (3)

$$\sum_i (F_i dr_i - \dot{p}_i dr_i) = 0 \quad \text{(3)}$$

$$\sum_j Q_j dq_j - \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right] dq_i = 0$$

$$\sum_j Q_j dq_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_i} dq_i = 0$$

$$\sum_j \left[Q_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j}\right] dq_j = 0$$

$$Q_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j} = 0$$

$$Q_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = 0 \quad \text{--- (11)} \quad j = 1, 2, 3$$

equation (11) is called Lagrange's equation of motion

Put $Q_j = -\frac{\partial V}{\partial q_j}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial}{\partial q_i}(T - V) = 0$$

(3)

where $L = T - V$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{——} (12)$$

equation (12) is called lagrange's equation

The Essence Of Lagrangian Dynamics

Although Hamilton principle was used in the formulation of Lagrange's equation using Variational techniques. This is not to say that in the least that hamilton statement was derived earlier than Lagrange's equation of motion. The truth is that Lagrange's equation was derived prior to the statement of hamilton.

(1) Since Energy is a Scalar Quantity, the Lagrange's function for a system is invariant to Co-ordinate transformation. Indeed Such transformation are not restricted to be between various orthogonal Co-ordinate System in ordinary Space Co-ordinate.

2) In Some case, it may not be possible to state explicitly all the forces acting on a body, when as it is still possible to give expression for the kinetic and potential Energy. It is just the fact that make hamilton principle useful for Quantum mechanical system in which the force are Sometime not known but the Energies are know.

Using variational technique show that Euler-Lagrange's equation can be written as

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

Solution

$$L = T - V$$

let $L = f(q_i, \dot{q}_i, t)$

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \cancel{\frac{\partial L}{\partial t}}^{0} dt$$

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \quad \text{—— (1)}$$

$$I = \int_{t_1}^{t_2} L\,dt$$

$$dI = \int_{t_1}^{t_2} dL\,dt \quad , dI = 0$$

$$\int_{t_1}^{t_2} dL\,dt = 0 \quad \text{—— (2)}$$

putting equation 1 into 2

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\right) dt = 0 \quad \text{—— (3)}$$

from equation (3) consider

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i$$ by using integrating part.

(5) let $u = \frac{\partial L}{\partial q}$ $dv = d\dot{q}_i$

$$du = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \qquad V = \int_{t_1}^{t_2} d\dot{q}_j$$

$$\int_{t_1}^{t_2} u\,dv = uv - \int_{t_1}^{t_2} v\,du.$$

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j = \frac{\partial L}{\partial \dot{q}_j}\int_{t_1}^{t_2} d\dot{q}_j - \int_{t_1}^{t_2} d\dot{q}_j \cdot \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$$

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j = \frac{\partial L}{\partial \dot{q}_j}\int_{t_1}^{t_2} d\dot{q}_j - \int_{t_1}^{t} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) d\dot{q}_j$$

But $d\dot{q}_j(t_2) - d\dot{q}_j(t_1) = 0$

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j = -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) d\dot{q}_j \quad \text{—— (4)}$$

Putting equation (4) into (3)

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_j} dq_j - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) dq_j \right) dt = 0$$

$$\int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right) dq_j\right] dt = 0$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad \text{—— (5)}$$

proved

Equation (5) is called Euler equation of motion

(6)

(2bi) State the Condition under which Lagrange's and Newton's equation are equivalent.

Solution

Let Consider the equation below.

$$\frac{\partial L}{\partial x_i} - \frac{\partial}{dt}\left(\frac{\partial L}{\dot{x}_i}\right) = 0 \quad (1)$$

from the lagrangian's equation

$$L = T - V$$

$$\frac{\partial (T-V)}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{x}_i}\right) = 0$$

$$\frac{\partial T}{\partial x_i} - \frac{\partial V}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i}\right) = 0$$

But $\frac{\partial T}{\partial x_i} = 0$ or $\frac{\partial V}{\partial \dot{x}_i}$

$$0 - \frac{\partial V}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - 0 = 0$$

$$-\frac{\partial V}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = 0$$

$$-\frac{\partial V}{\partial x_i} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)$$

where, $F = -\frac{\partial V}{\partial x_i}$, $T = \frac{1}{2}mv^2$

$$F = \frac{d}{dt}\left(\frac{\partial \frac{1}{2}mv^2}{\partial \dot{x}_i}\right)$$

$$F = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}} \frac{1}{2}m\dot{x}_i^2\right)$$

$$F = \frac{d}{dt}[m\dot{x}]$$

$$F = m\left[\frac{d}{dt}\dot{x}\right]$$

$$F = mv$$

$$f = \dot{p} \text{ proved}$$

(7)

Conservation Of energy

(Q5b) Prove the condition that leads to $\frac{dG}{dt} = \frac{dH}{dt} = 0$ and hence state the two condition necessary for $H = G$. If $H \neq G$ is the energy conserved? Illustrate.

Sol

Let $L = L(q, \dot{q}, t)$

$$dL = \frac{\partial L}{\partial q} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \cancel{\frac{\partial L}{\partial t} dt} \quad \text{—— (1)}$$

Take the summation of equation (1)

$$dL = \sum_j \frac{\partial L}{\partial q_i} dq_i + \sum_j \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \quad \text{—— (1*)}$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_j \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \quad \text{—— (2)}$$

from euler equation of motion

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \quad \text{—— (3)}$$

from equation (1*)

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial q}\right)\dot{q}_i$$

$$\frac{\partial L}{\partial t} - \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial q}\right)\dot{q}_i = 0$$

$$\frac{d}{dt}\left(L - \sum_j \left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\right) = 0$$

$$\frac{d}{dt}\left(L - \sum_j \left(\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j\right)\right) = 0 \quad \text{--- (4)}$$

But $\sum_j \left(\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j\right) = 2T$ --- (5)

putting (5) into 4

$$\frac{d}{dt}(L - 2T) = -H \text{ Constant} = 0$$

where $L = T - V$

$$L - 2T = -H$$

$$T - V - 2T = -H$$

$$-V - T = -H$$

$$T + V = H$$

$$H = T + V = E = 0 \text{ Constant.}$$

$$H = E = 0$$

$$\frac{dH}{dt} = \frac{dE}{dt} = 0 \quad \text{--- (6)}$$

proved

equation (7) is the total energy in constant motion
H is called the Hamiltonian of a System.

The hamiltonian is equal to the total energy only if the following are true :-

(i) The equation of transformation connecting the rectangular and generalized coordinate must be independent of the time.

ii) The potential energy must be velocity.

(9)

(Q3b) Derive Hamilton's equation of motion and hence State the condition under which the Hamiltonian Conserved.

Solution

Let consider the following function of the generalized Coordinate

$$H = \sum_j \dot{q}_j P_j - L \quad \text{—(1)}$$

But $\sum_j \dot{q}_j P_j = \sum \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j = \sum_j \frac{\partial T}{\partial \dot{q}_j} \cdot \dot{q}_j = 2T$ —(2)

Putting (2) in to equation (1)

where $L = T - V$

$$H = 2T - L$$
$$H = 2T - (T - V)$$
$$H = 2T - T + V$$
$$H = T + V$$

$$H(P, q, t) = \sum_j \dot{q}_j P_j - L(q, \dot{q}, t)$$

$$H = \sum \dot{q}_j P_j - L(q_j \dot{q}_j t)$$

$$dH = \sum \dot{q}_j dp + \cancel{\sum P_j d\dot{q}_j} - \sum \frac{\partial L}{\partial q_j} dq_j - \cancel{\sum \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j} - \sum \frac{\partial L}{\partial t} dt \quad \text{—(3)}$$

$$dH = \sum \dot{q}_j dp - \sum \frac{\partial L}{\partial q_j} dq_j - \sum \frac{\partial L}{\partial t} dt \quad \text{—(4)}$$

But $H = H(P_j, q_j, t)$

$$dH = \sum \frac{\partial H}{\partial P_j} dP_j + \sum \frac{\partial H}{\partial q_j} dq_j + \sum \frac{\partial H}{\partial t} dt \quad \text{—(5)}$$

Comparing (4) & (5)

$$\cancel{\sum} \frac{\partial H}{\partial P_i} \cancel{dP_i} = \cancel{\sum} \dot{q}_i \cancel{dP}$$

$$\dot{q}_i = \frac{\partial H}{\partial P_i} \quad \text{—(6)}$$

$$-\cancel{\sum} \frac{\partial L}{\partial q_j} \cancel{dq_j} = \cancel{\sum} \frac{\partial H}{\partial q_j} \cancel{dq_j}$$

$$-\frac{\partial L}{\partial q_j} = \frac{\partial H}{\partial q_j} \quad \text{—(7)}$$

$$-\cancel{\sum} \frac{\partial L}{\partial t} \cancel{dt} = \cancel{\sum} \frac{\partial H}{\partial t} \cancel{dt}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad \text{— 8.}$$

$$\Rightarrow \dot{q}_i = \frac{\partial H}{\partial q_i} \quad \text{— 6}$$

$$-\frac{\partial L}{\partial q_j} = \frac{\partial H}{\partial q_j} \quad \text{—(7)}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad \text{—(8)}$$

equation 6, 7 & 8 are Hamilton function of motion or Cononical equation of motion.

(2bi) State the conditions under which Lagrange's undetermined multiplier method Can be used with holonomic System.

Sol

Let Consider Consider holonomic Constraint

$$g(q_1, q_2, \ldots q_i) = 0 \quad \text{— 1}$$

$$\frac{\partial g}{\partial q_1} dq_i + \frac{\partial g}{\partial q_2} dq_2 + \ldots \frac{\partial g}{\partial q_i} dq_i = 0$$

$$\sum_{j=1}^{N} \frac{\partial g}{\partial q_j} dq_j = 0 \quad \text{—(2)}$$

Let $\frac{\partial g}{\partial q_j} = h_j$ —— (3)

Putting (3) into

$$\sum_{j=1}^{N} h_j dq_j = 0 \quad \text{—— (4)}$$

multiplying equation 4 by parameter λ

$$\sum_{j=1}^{N} \lambda h_j dq_j = 0 \; ; \; \sum_{j}^{N} dq_j \neq 0 , \; \lambda h_j = 0 \quad \text{—— (5)}$$

$$\int_{t_1}^{t_2} \sum_{j=1}^{N} \lambda h_j dq_j \, dt = 0 \quad \text{—— (6)}$$

from the integral of Varational integral equation.

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \right] dt = 0 \quad \text{—— (7)}$$

Adding 6 & 7.

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \right] dt + \int_{t_1}^{t_2} \sum_{j=1}^{N} \lambda h_j dq_j \, dt = 0$$

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_{j=1}^{N} \lambda h_j dq_j \right] dt = 0 \quad \text{—— (8)}$$

Applying fundamental Lemma Calculus.

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_{j=1}^{N} \lambda h_j dq_j = 0 \quad \text{—— (9)} \quad \text{if} \sum_{j=1}^{N} \partial q_j \neq 0$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \lambda h_j = 0 \quad \text{—— 10}$$

equation (10) is the Lagranges Equation of motion of under undetermined multiplier method.

(12)

(Q3c) Construct Hamilton's equation of motion for a simple harmonic oscillator in one dimension and determine its frequency if the mass of the oscillator is 2kg and the spring constant is 200Nm^{-1}

Solution

let $T = \frac{1}{2} m\dot{x}^2$; $V = \frac{1}{2} kx^2$ —— (1)

from Lagrange's equation

$L = T - V$ —— (2)

Puting equation into 2

$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$ —— (3)

from the momentum $P = \frac{\partial L}{\partial \dot{x}}$ —— (4)

putting 3 into 4

$p = \frac{\partial}{\partial \dot{x}}(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2)$

$p = m\dot{x}$

But $\dot{x} = \frac{P}{m}$

from Hamilton equation

$H = T + V$ —— (5)

But $T = \frac{p^2}{2m}$, $V = \frac{1}{2} kx^2$

$H = \frac{p^2}{2m} + \frac{1}{2} kx^2$

$\dot{x} = \frac{\partial H}{\partial P} = \frac{\partial}{\partial P}(\frac{p^2}{2m} + \frac{1}{2} kx^2)$

$\dot{x} = \frac{2P}{2m} + 0$

$\dot{x} = \frac{P}{m}$

$P = m\dot{x}$ —— (6)

$\frac{\partial H}{\partial x} = -\dot{p}$

$-\dot{p} = \frac{\partial}{\partial x}(\frac{p^2}{2m} + \frac{kx^2}{2})$

$-\dot{P} = 0 + \frac{2kx}{2}$

$-\dot{p} = kx$ —— 8

Putting equation 6 into 8

$-m\ddot{x} = kx$

$kx + m\ddot{x} = 0$ —— (9)

equation 9 is called Hamilton equation of motion of one dimension simple harmonic oscillators

13

Show that $\sum_i \dot{q} \frac{\partial T}{\partial \dot{q}} = 2\sum a_{jk} \dot{q}_j \dot{q}_k = 2T$

Solution

Consider a system of particles of masses m_i and position vector r_i the kinetic energy of the system is given by

$$T = \frac{1}{2}\sum_i m_i \dot{r}_i^2 \quad \text{——— (1)}$$

$$r_i = r_i(q_1, q_2, q_3 \text{---} q_j, t)$$

$$\dot{r}_i = \sum_k \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t} \quad \text{——— (2)}$$

putting equation 2 into 1

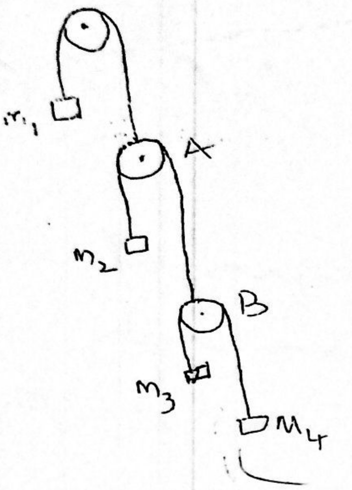
$$T = \frac{1}{2}\sum_i m_i \left[\sum_j \left(\frac{\partial r}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right)\sum_k \left(\frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}\right)\right]$$

$$T = \frac{1}{2}\sum_i m_i \left[\sum_j\sum_k \frac{\partial r_i}{\partial q_j}\cdot\frac{\partial r_i}{\partial q_k}\cdot\dot{q}_j\cdot\dot{q}_k + \frac{\partial r_i}{\partial q_j}\cdot\frac{\partial r_i}{\partial t}\dot{q}_j + \right.$$

$$\left. \frac{\partial r_i}{\partial t}\cdot\frac{\partial r_i}{\partial q_k}\dot{q}_k + \left(\frac{\partial r_i}{\partial t}\right)^2\right]$$

TUTORIAL QUESTIONS

PHY 301



$f(x) = \cos x$

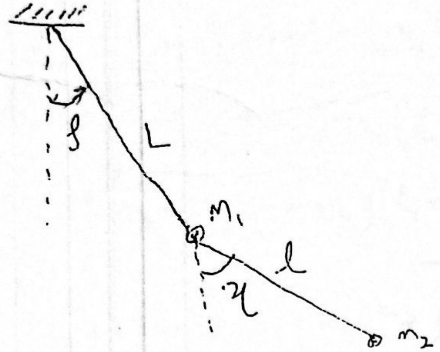
$f(-x) = \cos(-x) = \cos x$

Q1 The figure above refers. Given that the vertical part of the strings (i.e. m_1 to A, m_2 to B and m_3 to m_4) have lengths l_1, l_2 and l_3 respectively, find the Lagrangian density function, L and the equation of motion in each of the independent coordinates. The pulleys are frictionless and the strings inextensible. If $m_1 = m_4$ and $m_2 = m_3 = 2m$, find the acceleration of the independent coordinates.

$m_1 = m_2 = m_3 = m_4 = 2m$

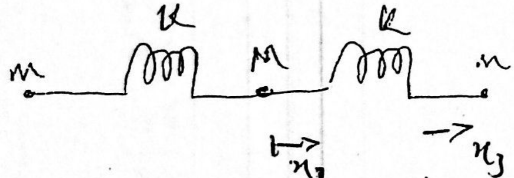
Q2 Two mass points m_1 and m_2 are connected by string passing through a hole on a smooth table so that m_1 ~~and m_2 are~~ is on the table and m_2 hangs suspended. Assuming that m_2 moves only on the vertical line, derive and solve the resulting equations of motion using Lagrangian method. Assuming the motion of m_1 is constant, calculate the ~~acceleration~~ due to gravity, g if ~~m_1~~ $m_1 = m_2$ and the velocity and the length of the first mass are $10 ms^{-1}$ and $10 m$ respectively.

Q3



Determine the Lagrangian density function for the system above and hence generalize to the case $m_1 = m_2 = m$ and $L = 2l$

Q4

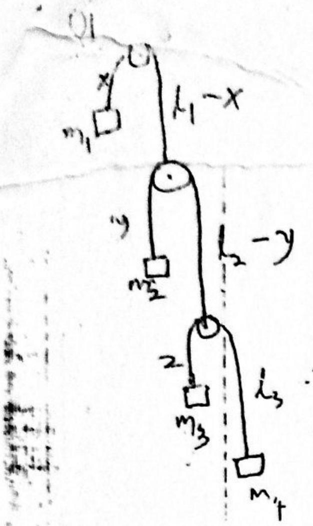


Determine the characteristic frequencies for the system above

$$|V - \omega^2 T|$$

Solution To Tutorial Questions

Q1



$$T = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 (\dot{l}_1 - \dot{x} + \dot{y})^2 + \tfrac{1}{2} m_3 (\dot{l}_1 - \dot{x} + \dot{y} + \dot{z})^2 + \tfrac{1}{2} m_4 (\dot{l}_1 - \dot{x} + \dot{l}_2 - \dot{y} + \dot{l}_3 - \dot{z})^2$$

$$V = m_1 g x + m_2 g (l_1 - x + y) + m_3 g (l_1 - x + y + z) + m_4 g (l_1 - x + l_2 - y + l_3 - z)$$

$$L = T - V = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 (\dot{x} - \dot{y})^2 + \tfrac{1}{2} m_3 (\dot{x} - \dot{y} - \dot{z})^2 + \tfrac{1}{2} m_4 (\dot{x} + \dot{y} + \dot{z})^2 - (m_1 - m_2 - m_3 - m_4) g x - (m_2 + m_3 - m_4) g y - (m_3 - m_4) g z$$

The equations of motion can be obtained from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

$$\ddot{x}(m_1 + m_2 + m_3 + m_4) + \ddot{y}(m_4 - m_2 - m_3) + \ddot{z}(m_4 - m_3) + (m_1 - m_2 - m_3 - m_4) g = 0$$

$$\ddot{x}(m_4 - m_2 - m_3) + \ddot{y}(m_2 + m_3 + m_4) + \ddot{z}(m_4 - m_3) + (m_2 + m_3 - m_4) g = 0$$

$$\ddot{x}(m_4 - m_3) + \ddot{y}(m_3 + m_4) + \ddot{z}(m_4 - m_3) + (m_3 - m_4) g = 0$$

For $m_1 = m_4 = 2m$ and $m_2 = m_3 = m$

$$6\ddot{x} + \ddot{z} = 2g$$
$$4\ddot{y} + \ddot{z} = 0$$
$$\ddot{x} + 3\ddot{y} + \ddot{z} = g$$

Now

$$4\ddot{y} = -\ddot{z}$$
$$\ddot{y} = -(\ddot{z}/4)$$

Solving simultaneously we have

$$6\ddot{x} + \ddot{z} = 2g \quad \cdots (1)$$
$$4\ddot{x} + \ddot{z} = 4g \quad \cdots (2)$$

Subtracting (2) from (1), we have

$$2\ddot{x} = -2g$$

$$\therefore \ddot{x} = -g$$

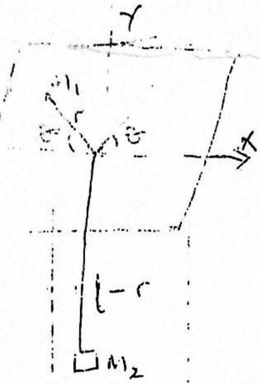
$$\ddot{z} = 8g$$

and

$$\ddot{y} = -2g$$

$$\ddot{x} = -g$$
$$\ddot{y} = -2g$$
$$\ddot{z} = 8g$$

Q3



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\dot{x}^2 = (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})^2$$

$$\dot{y}^2 = (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})^2.$$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$T = \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m_2 \dot{r}^2$$

$$V = - m_2 g (l - r)$$

$$L = T - V = \frac{1}{2} m_1 (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m_2 \dot{r}^2 + m_2 g l - m_2 g r \quad (1)$$

The equations of motion are thus

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \end{aligned}\right\} (2)$$

Then we have

$$(m_1 + m_2)\ddot{r} = m_1 r\dot{\theta}^2 - m_2 g \quad (3a)$$

$$m_1 r^2 \dot{\theta} = 0 = \text{constant} \quad (3b)$$

Equation (3b) merely expresses the conservation of angular momentum.

(ii) The motion m_2 balances m_1 whenever $\ddot{r} = 0$

In this case

$$m_1 r\dot{\theta}^2 = m_2 g$$

However

$$r\dot{\theta}^2 = \frac{v^2}{r}$$

$$m_1 \frac{v_1^2}{r} = m_2 g$$

$$g' = \frac{m_1 v_1^2}{m_2 r} = \frac{v_1^2}{r}$$

Q3

$X = L \sin \rho$

$Y = L \cos \rho$

$x = L \sin \rho + l \sin \eta$

$y = L \cos \rho + l \cos \eta$

$\dot X = L \cos \rho \dot\rho \qquad \dot x = L \cos\rho \dot\rho + l \cos \eta \dot\eta$

$\dot Y = - L \sin \rho \dot\rho \qquad \dot y = -(L \sin\rho \dot\rho + l \sin\eta \dot\eta)$

$\dot X^2 = L^2 \cos^2 \rho \dot\rho^2 \qquad \dot x^2 = L^2 \cos^2 \rho \dot\rho^2 + 2Ll \cos\rho\dot\rho \cos\eta\dot\eta + l^2 \cos^2\eta \dot\eta^2$

$\dot Y^2 = L^2 \sin^2 \rho \dot\rho^2 \qquad \dot y^2 = L^2 \sin^2 \rho \dot\rho^2 + l^2 \sin^2 \eta \dot\eta^2 + 2Ll \sin\rho\dot\rho \sin\eta\dot\eta$

$$T = \tfrac{1}{2} m_1 (\dot X^2 + \dot Y^2) + \tfrac{1}{2} m_2 (\dot x^2 + \dot y^2)$$

$$= \tfrac{1}{2} m_1 L^2 \dot\rho^2 + \tfrac{1}{2} m_2 L^2 \dot\rho^2 + \tfrac{1}{2} m_2 l^2 \dot\eta^2 + m_2 L l \cos(\rho - \eta) \dot\rho \dot\eta$$

$$V = - m_1 g Y - m_2 g y = -(m_1 + m_2) g L \cos\rho - m g l \cos \eta$$

$$L = T - V = \tfrac{1}{2}(m_1 + m_2) L^2 \dot\rho^2 + \tfrac{1}{2} m_2 l^2 \dot\eta^2 + \underline{m_2 L l \cos(\rho - \eta) \dot\rho \dot\eta}$$

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\rho}\right) - \frac{\partial L}{\partial \rho} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\eta}\right) - \frac{\partial L}{\partial \eta} = 0$$

which yields

$$\ddot\rho + \frac{g}{L}\rho = - \frac{m_2}{m_1 + m_2} \frac{l}{L} \ddot\eta$$

$$\ddot\rho = -\left(\frac{l}{L}\ddot\eta + \frac{g}{L}\eta\right)$$

for $m_1 = m_2 = m$, $L = 2l$

$$\ddot\rho + g/2l\, \rho = -1/4\, \ddot\eta$$

$$\ddot\rho \quad \left(\tfrac{1}{2}\ddot\eta + \frac{g}{2l}\eta\right).$$