CHAPTER

3 Potentials

3.1 ■ LAPLACE'S EQUATION

3.1.1 ■ Introduction

The primary task of electrostatics is to find the electric field of a given stationary charge distribution. In principle, this purpose is accomplished by Coulomb's law, in the form of Eq. 2.8:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\rho(\mathbf{r}')\,d\tau'. \tag{3.1}$$

Unfortunately, integrals of this type can be difficult to calculate for any but the simplest charge configurations. Occasionally we can get around this by exploiting symmetry and using Gauss's law, but ordinarily the best strategy is first to calculate the *potential,* V, which is given by the somewhat more tractable Eq. 2.29:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\imath}\rho(\mathbf{r}')\,d\tau'. \tag{3.2}$$

Still, even *this* integral is often too tough to handle analytically. Moreover, in problems involving conductors ρ itself may not be known in advance; since charge is free to move around, the only thing we control directly is the *total* charge (or perhaps the potential) of each conductor.

In such cases, it is fruitful to recast the problem in differential form, using Poisson's equation (2.24),

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho, \tag{3.3}$$

which, together with appropriate boundary conditions, is equivalent to Eq. 3.2. Very often, in fact, we are interested in finding the potential in a region where $\rho = 0$. (If $\rho = 0$ *everywhere,* of course, then $V = 0$, and there is nothing further to say—that's not what I mean. There may be plenty of charge *elsewhere,* but we're confining our attention to places where there is no charge.) In this case, Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0, \tag{3.4}$$

or, written out in Cartesian coordinates,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{3.5}$$

This formula is so fundamental to the subject that one might almost say electrostatics *is* the study of Laplace's equation. At the same time, it is a ubiquitous equation, appearing in such diverse branches of physics as gravitation and magnetism, the theory of heat, and the study of soap bubbles. In mathematics, it plays a major role in analytic function theory. To get a feel for Laplace's equation and its solutions (which are called **harmonic functions**), we shall begin with the one- and two-dimensional versions, which are easier to picture, and illustrate all the essential properties of the three-dimensional case.

3.1.2 ■ Laplace's Equation in One Dimension

Suppose V depends on only one variable, x. Then Laplace's equation becomes

$$\frac{d^2 V}{dx^2} = 0.$$

The general solution is

$$V(x) = mx + b, \tag{3.6}$$

the equation for a straight line. It contains two undetermined constants (m and b), as is appropriate for a second-order (ordinary) differential equation. They are fixed, in any particular case, by the boundary conditions of that problem. For instance, it might be specified that $V = 4$ at $x = 1$, and $V = 0$ at $x = 5$. In that case, $m = -1$ and $b = 5$, so $V = -x + 5$ (see Fig. 3.1).

I want to call your attention to two features of this result; they may seem silly and obvious in one dimension, where I can write down the general solution explicitly, but the analogs in two and three dimensions are powerful and by no means obvious:

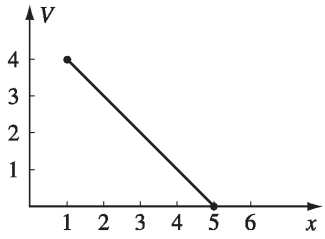


FIGURE 3.1

1. $V(x)$ is the *average* of $V(x+a)$ and $V(x-a)$, for any a:

$$V(x) = \frac{1}{2}[V(x+a) + V(x-a)].$$

Laplace's equation is a kind of averaging instruction; it tells you to assign to the point x the average of the values to the left and to the right of x. Solutions to Laplace's equation are, in this sense, *as boring as they could possibly be,* and yet fit the end points properly.

2. Laplace's equation tolerates *no local maxima or minima;* extreme values of V must occur at the end points. Actually, this is a consequence of (1), for if there *were* a local maximum, V would be greater at that point than on either side, and therefore could not be the average. (Ordinarily, you expect the second derivative to be negative at a maximum and positive at a minimum. Since Laplace's equation requires, on the contrary, that the second derivative is zero, it seems reasonable that solutions should exhibit no extrema. However, this is not a *proof,* since there exist functions that have maxima and minima at points where the second derivative vanishes: x^4, for example, has such a minimum at the point $x = 0$.)

3.1.3 ■ Laplace's Equation in Two Dimensions

If V depends on two variables, Laplace's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

This is no longer an *ordinary* differential equation (that is, one involving ordinary derivatives only); it is a *partial* differential equation. As a consequence, some of the simple rules you may be familiar with do not apply. For instance, the general solution to this equation doesn't contain just two arbitrary constants—or, for that matter, *any* finite number—despite the fact that it's a second-order equation. Indeed, one cannot write down a "general solution" (at least, not in a closed form like Eq. 3.6). Nevertheless, it is possible to deduce certain properties common to all solutions.

It may help to have a physical example in mind. Picture a thin rubber sheet (or a soap film) stretched over some support. For definiteness, suppose you take a cardboard box, cut a wavy line all the way around, and remove the top part (Fig. 3.2). Now glue a tightly stretched rubber membrane over the box, so that it fits like a drum head (it won't be a *flat* drumhead, of course, unless you chose to cut the edges off straight). Now, if you lay out coordinates (x, y) on the bottom of the box, the height $V(x, y)$ of the sheet above the point (x, y) will satisfy Laplace's

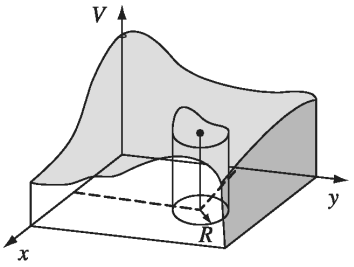


FIGURE 3.2

equation.[1] (The one-dimensional analog would be a rubber band stretched between two points. Of course, it would form a straight line.)

Harmonic functions in two dimensions have the same properties we noted in one dimension:

1. The value of V at a point (x, y) is the average of those *around* the point. More precisely, if you draw a circle of any radius R about the point (x, y), the average value of V on the circle is equal to the value at the center:

$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V \, dl.$$

(This, incidentally, suggests the **method of relaxation**, on which computer solutions to Laplace's equation are based: Starting with specified values for V at the boundary, and reasonable guesses for V on a grid of interior points, the first pass reassigns to each point the average of its nearest neighbors. The second pass repeats the process, using the corrected values, and so on. After a few iterations, the numbers begin to settle down, so that subsequent passes produce negligible changes, and a numerical solution to Laplace's equation, with the given boundary values, has been achieved.)[2]

2. V has no local maxima or minima; all extrema occur at the boundaries. (As before, this follows from (1).) Again, Laplace's equation picks the most featureless function possible, consistent with the boundary conditions: no hills, no valleys, just the smoothest conceivable surface. For instance, if you put a ping-pong ball on the stretched rubber sheet of Fig. 3.2, it will

[1] Actually, the equation satisfied by a rubber sheet is

$$\frac{\partial}{\partial x}\left(g\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(g\frac{\partial V}{\partial y}\right) = 0, \quad \text{where } g = \left[1 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right]^{-1/2};$$

it reduces (approximately) to Laplace's equation as long as the surface does not deviate too radically from a plane.

[2] See, for example, E. M. Purcell, *Electricity and Magnetism,* 2nd ed. (New York: McGraw-Hill, 1985), problem 3.30.

roll over to one side and fall off—it will not find a "pocket" somewhere to settle into, for Laplace's equation allows no such dents in the surface. From a geometrical point of view, just as a straight line is the shortest distance between two points, so a harmonic function in two dimensions minimizes the surface area spanning the given boundary line.

3.1.4 ■ Laplace's Equation in Three Dimensions

In three dimensions I can neither provide you with an explicit solution (as in one dimension) nor offer a suggestive physical example to guide your intuition (as I did in two dimensions). Nevertheless, the same two properties remain true, and this time I will sketch a proof.[3]

1. The value of V at point $\mathbf{r}$ is the average value of V over a spherical surface of radius R centered at $\mathbf{r}$:

$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V \, da.$$

2. As a consequence, V can have no local maxima or minima; the extreme values of V must occur at the boundaries. (For if V had a local maximum at $\mathbf{r}$, then by the very nature of maximum I could draw a sphere around $\mathbf{r}$ over which all values of V—and *a fortiori* the average—would be less than at $\mathbf{r}$.)

Proof. Let's begin by calculating the average potential over a spherical surface of radius R due to a *single* point charge q located outside the sphere. We may as well center the sphere at the origin and choose coordinates so that q lies on the z-axis (Fig. 3.3). The potential at a point on the surface is

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}},$$

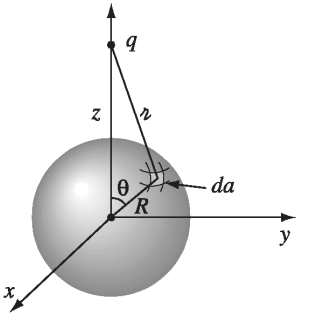


FIGURE 3.3

[3]For a proof that does not rely on Coulomb's law (only on Laplace's equation), see Prob. 3.37.

where

$$r^2 = z^2 + R^2 - 2zR\cos\theta,$$

so

$$\begin{aligned} V_{\text{ave}} &= \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int [z^2 + R^2 - 2zR\cos\theta]^{-1/2} R^2 \sin\theta\, d\theta\, d\phi \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\sqrt{z^2 + R^2 - 2zR\cos\theta}\,\Big|_0^\pi \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}[(z+R) - (z-R)] = \frac{1}{4\pi\epsilon_0}\frac{q}{z}. \end{aligned}$$

But this is precisely the potential due to q at the *center* of the sphere! By the superposition principle, the same goes for any *collection* of charges outside the sphere: their average potential over the sphere is equal to the net potential they produce at the center. □

Problem 3.1 Find the average potential over a spherical surface of radius R due to a point charge q located *inside* (same as above, in other words, only with $z < R$). (In this case, of course, Laplace's equation does not hold within the sphere.) Show that, in general,

$$V_{\text{ave}} = V_{\text{center}} + \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R},$$

where V_{center} is the potential at the center due to all the *external* charges, and Q_{enc} is the total enclosed charge.

Problem 3.2 In one sentence, justify **Earnshaw's Theorem:** *A charged particle cannot be held in a stable equilibrium by electrostatic forces alone.* As an example, consider the cubical arrangement of fixed charges in Fig. 3.4. It *looks,* off hand, as though a positive charge at the center would be suspended in midair, since it is repelled away from each corner. Where is the leak in this "electrostatic bottle"? [To harness nuclear fusion as a practical energy source it is necessary to heat a plasma (soup of charged particles) to fantastic temperatures—so hot that contact would vaporize any ordinary pot. Earnshaw's theorem says that electrostatic containment is also out of the question. Fortunately, it *is* possible to confine a hot plasma magnetically.]

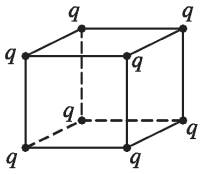


FIGURE 3.4

Problem 3.3 Find the general solution to Laplace's equation in spherical coordinates, for the case where V depends only on r. Do the same for cylindrical coordinates, assuming V depends only on s.

Problem 3.4

(a) Show that the average electric *field* over a spherical surface, due to charges outside the sphere, is the same as the field at the center.

(b) What is the average due to charges *inside* the sphere?

3.1.5 ■ Boundary Conditions and Uniqueness Theorems

Laplace's equation does not by itself determine V; in addition, suitable boundary conditions must be supplied. This raises a delicate question: What are appropriate boundary conditions, sufficient to determine the answer and yet not so strong as to generate inconsistencies? The one-dimensional case is easy, for here the general solution $V = mx + b$ contains two arbitrary constants, and we therefore require two boundary conditions. We might, for instance, specify the value of the function at each end, or we might give the value of the function and its derivative at one end, or the value at one end and the derivative at the other, and so on. But we cannot get away with *just* the value or *just* the derivative at *one* end—this is insufficient information. Nor would it do to specify the derivatives at both ends—this would either be redundant (if the two are equal) or inconsistent (if they are not).

In two or three dimensions we are confronted by a *partial* differential equation, and it is not so obvious what would constitute acceptable boundary conditions. Is the shape of a taut rubber membrane, for instance, uniquely determined by the frame over which it is stretched, or, like a canning jar lid, can it snap from one stable configuration to another? The answer, as I think your intuition would suggest, is that V *is* uniquely determined by its value at the boundary (canning jars evidently do not obey Laplace's equation). However, other boundary conditions can also be used (see Prob. 3.5). The *proof* that a proposed set of boundary conditions will suffice is usually presented in the form of a **uniqueness theorem.** There are many such theorems for electrostatics, all sharing the same basic format—I'll show you the two most useful ones.[4]

> **First uniqueness theorem:** The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$.

Proof. In Fig. 3.5 I have drawn such a region and its boundary. (There could also be "islands" inside, so long as V is given on all their surfaces; also, the outer

[4] I do not intend to prove the *existence* of solutions here—that's a much more difficult job. In context, the existence is generally clear on physical grounds.

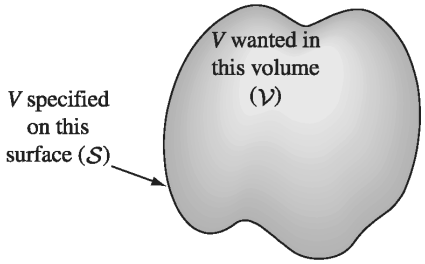


FIGURE 3.5

boundary could be at infinity, where V is ordinarily taken to be zero.) Suppose there were *two* solutions to Laplace's equation:

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0,$$

both of which assume the specified value on the surface. I want to prove that they must be equal. The trick is look at their *difference:*

$$V_3 \equiv V_1 - V_2.$$

This obeys Laplace's equation,

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0,$$

and it takes the value *zero* on all boundaries (since V_1 and V_2 are equal there). But Laplace's equation allows no local maxima or minima—all extrema occur on the boundaries. So the maximum and minimum of V_3 are both zero. Therefore V_3 must be zero everywhere, and hence

$$V_1 = V_2. \qquad \square$$

Example 3.1. Show that the potential is *constant* inside an enclosure completely surrounded by conducting material, provided there is no charge within the enclosure.

Solution

The potential on the cavity wall is some constant, V_0 (that's item (iv), in Sect. 2.5.1), so the potential inside is a function that satisfies Laplace's equation and has the constant value V_0 at the boundary. It doesn't take a genius to think of *one* solution to this problem: $V = V_0$ everywhere. The uniqueness theorem guarantees that this is the *only* solution. (It follows that the *field* inside an empty cavity is zero—the same result we found in Sect. 2.5.2 on rather different grounds.)

The uniqueness theorem is a license to your imagination. It doesn't matter *how* you come by your solution; if (a) it satisfies Laplace's equation and (b) it has

the correct value on the boundaries, then it's *right.* You'll see the power of this argument when we come to the method of images.

Incidentally, it is easy to improve on the first uniqueness theorem: I assumed there was no charge inside the region in question, so the potential obeyed Laplace's equation, but we may as well throw in some charge (in which case V obeys Poisson's equation). The argument is the same, only this time

$$\nabla^2 V_1 = -\frac{1}{\epsilon_0}\rho, \qquad \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho,$$

so

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho + \frac{1}{\epsilon_0}\rho = 0.$$

Once again the *difference* ($V_3 \equiv V_1 - V_2$) satisfies Laplace's equation and has the value zero on all boundaries, so $V_3 = 0$ and hence $V_1 = V_2$.

Corollary: The potential in a volume $\mathcal{V}$ is uniquely determined if (a) the charge density throughout the region, and (b) the value of V on all boundaries, are specified.

3.1.6 ■ Conductors and the Second Uniqueness Theorem

The *simplest* way to set the boundary conditions for an electrostatic problem is to specify the value of V on all surfaces surrounding the region of interest. And this situation often occurs in practice: In the laboratory, we have conductors connected to batteries, which maintain a given potential, or to **ground**, which is the experimentalist's word for $V = 0$. However, there are other circumstances in which we do not know the *potential* at the boundary, but rather the *charges* on various conducting surfaces. Suppose I put charge Q_a on the first conductor, Q_b on the second, and so on—I'm not telling you how the charge distributes itself over each conducting surface, because as soon as I put it on, it moves around in a way I do not control. And for good measure, let's say there is some specified charge density ρ in the region between the conductors. Is the electric field now uniquely determined? Or are there perhaps a number of different ways the charges could arrange themselves on their respective conductors, each leading to a different field?

Second uniqueness theorem: In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the *total charge* on each conductor is given (Fig. 3.6). (The region as a whole can be bounded by another conductor, or else unbounded.)

Proof. Suppose there are *two* fields satisfying the conditions of the problem. Both obey Gauss's law in differential form in the space between the conductors:

$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\epsilon_0}\rho, \qquad \nabla \cdot \mathbf{E}_2 = \frac{1}{\epsilon_0}\rho.$$

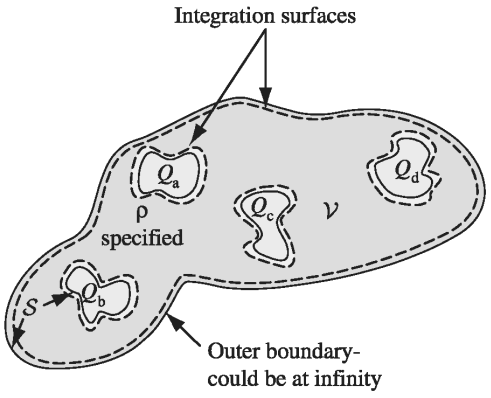


FIGURE 3.6

And both obey Gauss's law in integral form for a Gaussian surface enclosing each conductor:

$$\oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_i, \qquad \oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_i.$$

Likewise, for the outer boundary (whether this is just inside an enclosing conductor or at infinity),

$$\oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{tot}}, \qquad \oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{tot}}.$$

As before, we examine the difference

$$\mathbf{E}_3 \equiv \mathbf{E}_1 - \mathbf{E}_2,$$

which obeys

$$\boldsymbol{\nabla} \cdot \mathbf{E}_3 = 0 \tag{3.7}$$

in the region between the conductors, and

$$\oint \mathbf{E}_3 \cdot d\mathbf{a} = 0 \tag{3.8}$$

over each boundary surface.

Now there is one final piece of information we must exploit: Although we do not know how the charge Q_i distributes itself over the ith conductor, we *do* know that each conductor is an equipotential, and hence V_3 is a *constant* (not

necessarily the *same* constant) over each conducting surface. (It need not be *zero,* for the potentials V_1 and V_2 may not be equal—all we know for sure is that *both* are *constant* over any given conductor.) Next comes a trick. Invoking product rule number 5 (inside front cover), we find that

$$\nabla \cdot (V_3 \mathbf{E}_3) = V_3(\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot (\nabla V_3) = -(E_3)^2.$$

Here I have used Eq. 3.7, and $\mathbf{E}_3 = -\nabla V_3$. Integrating this over $\mathcal{V}$, and applying the divergence theorem to the left side:

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{E}_3)\, d\tau = \oint_{\mathcal{S}} V_3 \mathbf{E}_3 \cdot d\mathbf{a} = -\int_{\mathcal{V}} (E_3)^2\, d\tau.$$

The surface integral covers all boundaries of the region in question—the conductors and outer boundary. Now V_3 is a constant over each surface (if the outer boundary is infinity, $V_3 = 0$ there), so it comes outside each integral, and what remains is zero, according to Eq. 3.8. Therefore,

$$\int_{\mathcal{V}} (E_3)^2\, d\tau = 0.$$

But this integrand is never negative; the only way the integral can vanish is if $E_3 = 0$ everywhere. Consequently, $\mathbf{E}_1 = \mathbf{E}_2$, and the theorem is proved. □

This proof was not easy, and there is a real danger that the theorem itself will seem more plausible to you than the proof. In case you think the second uniqueness theorem is "obvious," consider this example of Purcell's: Figure 3.7 shows a simple electrostatic configuration, consisting of four conductors with charges $\pm Q$, situated so that the plusses are near the minuses. It all looks very comfortable. Now, what happens if we join them in pairs, by tiny wires, as indicated in Fig. 3.8? Since the positive charges are very near negative charges (which is where they *like* to be) you might well guess that *nothing* will happen—the configuration looks stable.

Well, that sounds reasonable, but it's wrong. The configuration in Fig. 3.8 is *impossible.* For there are now effectively *two* conductors, and the total charge on each is *zero*. *One* possible way to distribute zero charge over these conductors is to have no accumulation of charge anywhere, and hence zero field

FIGURE 3.7

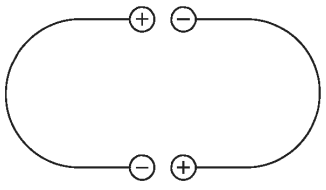


FIGURE 3.8

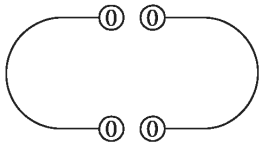


FIGURE 3.9

everywhere (Fig. 3.9). By the second uniqueness theorem, this must be *the* solution: The charge will flow down the tiny wires, canceling itself off.

Problem 3.5 Prove that the field is uniquely determined when the charge density ρ is given and *either* V *or* the normal derivative $\partial V/\partial n$ is specified on each boundary surface. Do not assume the boundaries are conductors, or that V is constant over any given surface.

Problem 3.6 A more elegant proof of the second uniqueness theorem uses Green's identity (Prob. 1.61c), with $T = U = V_3$. Supply the details.

3.2 ■ THE METHOD OF IMAGES

3.2.1 ■ The Classic Image Problem

Suppose a point charge q is held a distance d above an infinite grounded conducting plane (Fig. 3.10). *Question:* What is the potential in the region above the plane? It's not just $(1/4\pi\epsilon_0)q/\imath$, for q will induce a certain amount of negative charge on the nearby surface of the conductor; the total potential is due in part to q directly, and in part to this induced charge. But how can we possibly calculate the potential, when we don't know how much charge is induced or how it is distributed?

From a mathematical point of view, our problem is to solve Poisson's equation in the region $z > 0$, with a single point charge q at $(0, 0, d)$, subject to the boundary conditions:

1. $V = 0$ when $z = 0$ (since the conducting plane is grounded), and
2. $V \to 0$ far from the charge (that is, for $x^2 + y^2 + z^2 \gg d^2$).

The first uniqueness theorem (actually, its corollary) guarantees that there is only one function that meets these requirements. If by trick or clever guess we can discover such a function, it's got to be the answer.

Trick: Forget about the actual problem; we're going to study a *completely different* situation. This new configuration consists of *two* point charges, $+q$ at

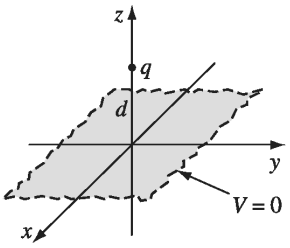


FIGURE 3.10

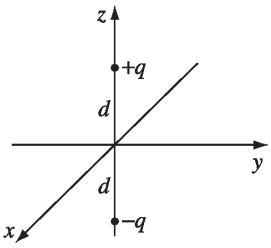


FIGURE 3.11

$(0, 0, d)$ and $-q$ at $(0, 0, -d)$, and *no* conducting plane (Fig. 3.11). For this configuration, I can easily write down the potential:

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]. \quad (3.9)$$

(The denominators represent the distances from (x, y, z) to the charges $+q$ and $-q$, respectively.) It follows that

1. $V = 0$ when $z = 0$,
2. $V \to 0$ for $x^2 + y^2 + z^2 \gg d^2$,

and the only charge in the region $z > 0$ is the point charge $+q$ at $(0, 0, d)$. But these are precisely the conditions of the original problem! Evidently the second configuration happens to produce exactly the same potential as the first configuration, in the "upper" region $z \geq 0$. (The "lower" region, $z < 0$, is completely different, but who cares? The upper part is all we need.) *Conclusion:* The potential of a point charge above an infinite grounded conductor is given by Eq. 3.9, for $z \geq 0$.

Notice the crucial role played by the uniqueness theorem in this argument: without it, no one would believe this solution, since it was obtained for a completely different charge distribution. But the uniqueness theorem certifies it: If it satisfies Poisson's equation in the region of interest, and assumes the correct value at the boundaries, then it must be right.

3.2.2 ■ Induced Surface Charge

Now that we know the potential, it is a straightforward matter to compute the surface charge σ induced on the conductor. According to Eq. 2.49,

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n},$$

where $\partial V/\partial n$ is the normal derivative of V at the surface. In this case the normal direction is the z direction, so

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0}.$$

From Eq. 3.9,

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\},$$

so[5]

$$\sigma(x, y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}. \tag{3.10}$$

As expected, the induced charge is negative (assuming q is positive) and greatest at $x = y = 0$.

While we're at it, let's compute the *total* induced charge

$$Q = \int \sigma \, da.$$

This integral, over the xy plane, could be done in Cartesian coordinates, with $da = dx\,dy$, but it's a little easier to use polar coordinates (r, ϕ), with $r^2 = x^2+y^2$ and $da = r\,dr\,d\phi$. Then

$$\sigma(r) = \frac{-qd}{2\pi(r^2+d^2)^{3/2}},$$

and

$$Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r\,dr\,d\phi = \frac{qd}{\sqrt{r^2+d^2}}\bigg|_0^{\infty} = -q. \tag{3.11}$$

The total charge induced on the plane is $-q$, as (with benefit of hindsight) you can perhaps convince yourself it *had* to be.

3.2.3 ■ Force and Energy

The charge q is attracted toward the plane, because of the negative induced charge. Let's calculate the force of attraction. Since the potential in the vicinity of q is the same as in the analog problem (the one with $+q$ and $-q$ but no conductor), so also is the field and, therefore, the force:

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{\mathbf{z}}. \tag{3.12}$$

[5]For an entirely different derivation of this result, see Prob. 3.38.

Beware: It is easy to get carried away, and assume that *everything* is the same in the two problems. Energy, however, is *not* the same. With the two point charges and no conductor, Eq. 2.42 gives

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}. \tag{3.13}$$

But for a single charge and conducting plane, the energy is *half* of this:

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}. \tag{3.14}$$

Why half? Think of the energy stored in the fields (Eq. 2.45):

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau.$$

In the first case, both the upper region ($z > 0$) and the lower region ($z < 0$) contribute—and by symmetry they contribute equally. But in the second case, only the upper region contains a nonzero field, and hence the energy is half as great.[6]

Of course, one could also determine the energy by calculating the work required to bring q in from infinity. The force required (to oppose the electrical force in Eq. 3.12) is $(1/4\pi\epsilon_0)(q^2/4z^2)\hat{\mathbf{z}}$, so

$$W = \int_\infty^d \mathbf{F}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_\infty^d \frac{q^2}{4z^2}\, dz$$
$$= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{4z}\right)\Big|_\infty^d = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}.$$

As I move q toward the conductor, I do work *only on* q. It is true that induced charge is moving in over the conductor, but this costs me nothing, since the whole conductor is at potential zero. By contrast, if I simultaneously bring in *two* point charges (with no conductor), I do work on *both* of them, and the total is (again) twice as great.

3.2.4 ■ Other Image Problems

The method just described is not limited to a single point charge; *any* stationary charge distribution near a grounded conducting plane can be treated in the same way, by introducing its mirror image—hence the name **method of images.** (Remember that the image charges have the *opposite sign;* this is what guarantees that the xy plane will be at potential zero.) There are also some exotic problems that can be handled in similar fashion; the nicest of these is the following.

[6]For a generalization of this result, see M. M. Taddei, T. N. C. Mendes, and C. Farina, *Eur. J. Phys.* **30**, 965 (2009), and Prob. 3.41b.

Example 3.2. A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R (Fig. 3.12). Find the potential outside the sphere.

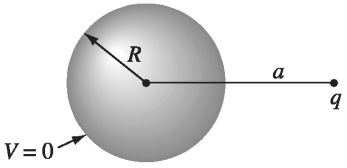


FIGURE 3.12

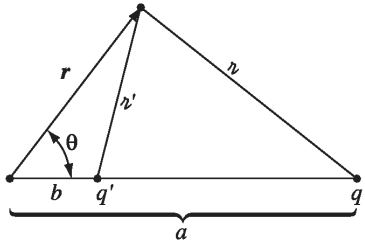


FIGURE 3.13

Solution

Examine the *completely different* configuration, consisting of the point charge q together with another point charge

$$q' = -\frac{R}{a}q, \tag{3.15}$$

placed a distance

$$b = \frac{R^2}{a} \tag{3.16}$$

to the right of the center of the sphere (Fig. 3.13). No conductor, now—just the two point charges. The potential of this configuration is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right), \tag{3.17}$$

where $\mathscr{r}$ and $\mathscr{r}'$ are the distances from q and q', respectively. Now, it happens (see Prob. 3.8) that this potential vanishes at all points on the sphere, and therefore fits the boundary conditions for our original problem, in the exterior region.[7]

Conclusion: Eq. 3.17 is the potential of a point charge near a grounded conducting sphere. (Notice that b is less than R, so the "image" charge q' is safely inside the sphere—*you cannot put image charges in the region where you are calculating* V; that would change ρ, and you'd be solving Poisson's equation with

[7]This solution is due to William Thomson (later Lord Kelvin), who published it in 1848, when he was just 24. It was apparently inspired by a theorem of Apollonius (200 BC) that says the locus of points with a fixed ratio of distances from two given points is a sphere. See J. C. Maxwell, "Treatise on Electricity and Magnetism, Vol. I," Dover, New York, p. 245. I thank Gabriel Karl for this interesting history.

the wrong source.) In particular, the force of attraction between the charge and the sphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2 Ra}{(a^2-R^2)^2}. \tag{3.18}$$

The method of images is delightfully simple ... when it works. But it is as much an art as a science, for you must somehow think up just the right "auxiliary" configuration, and for most shapes this is forbiddingly complicated, if not impossible.

Problem 3.7 Find the force on the charge $+q$ in Fig. 3.14. (The xy plane is a grounded conductor.)

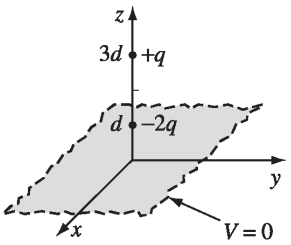


FIGURE 3.14

Problem 3.8

(a) Using the law of cosines, show that Eq. 3.17 can be written as follows:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right], \tag{3.19}$$

where r and θ are the usual spherical polar coordinates, with the z axis along the line through q. In this form, it is obvious that $V = 0$ on the sphere, $r = R$.

(b) Find the induced surface charge on the sphere, as a function of θ. Integrate this to get the total induced charge. (What *should* it be?)

(c) Calculate the energy of this configuration.

Problem 3.9 In Ex. 3.2 we assumed that the conducting sphere was grounded ($V = 0$). But with the addition of a second image charge, the same basic model will handle the case of a sphere at *any* potential V_0 (relative, of course, to infinity). What charge should you use, and where should you put it? Find the force of attraction between a point charge q and a *neutral* conducting sphere.

! **Problem 3.10** A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. (Let's say the wire runs parallel to the x-axis and directly above it, and the conducting plane is the xy plane.)

(a) Find the potential in the region above the plane. [*Hint:* Refer to Prob. 2.52.]

(b) Find the charge density σ induced on the conducting plane.

Problem 3.11 Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q, situated as shown in Fig. 3.15. Set up the image configuration, and calculate the potential in this region. What charges do you need, and where should they be located? What is the force on q? How much work did it take to bring q in from infinity? Suppose the planes met at some angle other than 90°; would you still be able to solve the problem by the method of images? If not, for what particular angles *does* the method work?

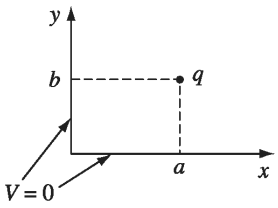


FIGURE 3.15

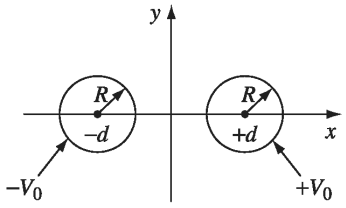


FIGURE 3.16

! **Problem 3.12** Two long, straight copper pipes, each of radius R, are held a distance $2d$ apart. One is at potential V_0, the other at $-V_0$ (Fig. 3.16). Find the potential everywhere. [*Hint:* Exploit the result of Prob. 2.52.]

3.3 ■ SEPARATION OF VARIABLES

In this section we shall attack Laplace's equation directly, using the method of **separation of variables**, which is the physicist's favorite tool for solving partial differential equations. The method is applicable in circumstances where the potential (V) or the charge density (σ) is specified on the boundaries of some region, and we are asked to find the potential in the interior. The basic strategy is very simple: *We look for solutions that are* products *of functions, each of which depends on only* one *of the coordinates*. The algebraic details, however, can be formidable, so I'm going to develop the method through a sequence of examples. We'll start with Cartesian coordinates and then do spherical coordinates (I'll leave the cylindrical case for you to tackle on your own, in Prob. 3.24).

3.3.1 ■ Cartesian Coordinates

Example 3.3. Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$ (Fig. 3.17). The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates, and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."

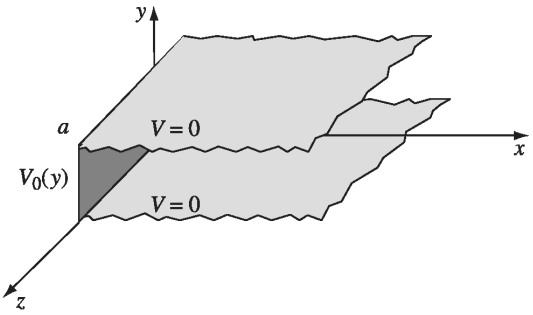


FIGURE 3.17

Solution

The configuration is independent of z, so this is really a *two*-dimensional problem. In mathematical terms, we must solve Laplace's equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \tag{3.20}$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0(y) \text{ when } x = 0, \\ \text{(iv)} & V \to 0 \text{ as } x \to \infty. \end{array}\right\} \tag{3.21}$$

(The latter, although not explicitly stated in the problem, is necessary on physical grounds: as you get farther and farther away from the "hot" strip at $x = 0$, the potential should drop to zero.) Since the potential is specified on all boundaries, the answer is uniquely determined.

The first step is to look for solutions in the form of products:

$$V(x, y) = X(x)Y(y). \tag{3.22}$$

On the face of it, this is an absurd restriction—the overwhelming majority of solutions to Laplace's equation do *not* have such a form. For example, $V(x, y) =$

$(5x + 6y)$ satisfies Eq. 3.20, but you can't express it as the product of a function x times a function y. Obviously, we're only going to get a tiny subset of all possible solutions by this means, and it would be a *miracle* if one of them happened to fit the boundary conditions of our problem ... But hang on, because the solutions we *do* get are very special, and it turns out that by pasting them together we can construct the general solution.

Anyway, putting Eq. 3.22 into Eq. 3.20, we obtain

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0.$$

The next step is to "separate the variables" (that is, collect all the x-dependence into one term and all the y-dependence into another). Typically, this is accomplished by dividing through by V:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0. \tag{3.23}$$

Here the first term depends only on x and the second only on y; in other words, we have an equation of the form

$$f(x) + g(y) = 0. \tag{3.24}$$

Now, there's only one way this could possibly be true: *f and g must both be constant.* For what if $f(x)$ *changed,* as you vary x—then if we held y fixed and fiddled with x, the sum $f(x) + g(y)$ would *change,* in violation of Eq. 3.24, which says it's always zero. (That's a simple but somehow rather elusive argument; don't accept it without due thought, because the whole method rides on it.)

It follows from Eq. 3.23, then, that

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \quad \text{with} \quad C_1 + C_2 = 0. \tag{3.25}$$

One of these constants is positive, the other negative (or perhaps both are zero). In general, one must investigate all the possibilities; however, in our particular problem we need C_1 positive and C_2 negative, for reasons that will appear in a moment. Thus

$$\frac{d^2X}{dx^2} = k^2X, \qquad \frac{d^2Y}{dy^2} = -k^2Y. \tag{3.26}$$

Notice what has happened: A *partial* differential equation (3.20) has been converted into two *ordinary* differential equations (3.26). The advantage of this is obvious—ordinary differential equations are a lot easier to solve. Indeed:

$$X(x) = Ae^{kx} + Be^{-kx}, \qquad Y(y) = C\sin ky + D\cos ky,$$

so

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky). \tag{3.27}$$

This is the appropriate separable solution to Laplace's equation; it remains to impose the boundary conditions, and see what they tell us about the constants. To begin at the end, condition (iv) requires that A equal zero.[8] Absorbing B into C and D, we are left with

$$V(x, y) = e^{-kx}(C \sin ky + D \cos ky).$$

Condition (i) now demands that D equal zero, so

$$V(x, y) = Ce^{-kx} \sin ky. \tag{3.28}$$

Meanwhile (ii) yields $\sin ka = 0$, from which it follows that

$$k = \frac{n\pi}{a}, \qquad (n = 1, 2, 3, \ldots). \tag{3.29}$$

(At this point you can see why I chose C_1 positive and C_2 negative: If X were sinusoidal, we could never arrange for it to go to zero at infinity, and if Y were exponential we could not make it vanish at both 0 and a. Incidentally, $n = 0$ is no good, for in that case the potential vanishes *everywhere*. And we have already excluded negative n's.)

That's as far as we can go, using separable solutions, and unless $V_0(y)$ just happens to have the form $\sin(n\pi y/a)$ for some integer n, we simply *can't fit* the final boundary condition at $x = 0$. But now comes the crucial step that redeems the method: Separation of variables has given us an *infinite family* of solutions (one for each n), and whereas none of them *by itself* satisfies the final boundary condition, it is possible to combine them in a way that *does*. Laplace's equation is *linear*, in the sense that if V_1, V_2, V_3, ... satisfy it, so does any **linear combination**, $V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \ldots$, where $\alpha_1, \alpha_2, \ldots$ are arbitrary constants. For

$$\nabla^2 V = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \ldots = 0\alpha_1 + 0\alpha_2 + \ldots = 0.$$

Exploiting this fact, we can patch together the separable solutions (Eq. 3.28) to construct a much more general solution:

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a). \tag{3.30}$$

This still satisfies three of the boundary conditions; the question is, can we (by astute choice of the coefficients C_n) fit the final boundary condition (iii)?

$$V(0, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y). \tag{3.31}$$

[8] I'm assuming k is positive, but this involves no loss of generality—negative k gives the same solution (Eq. 3.27), only with the constants shuffled ($A \leftrightarrow B$, $C \rightarrow -C$). Occasionally (though not in this example) $k = 0$ must also be included (see Prob. 3.54).

Well, you may recognize this sum—it's a **Fourier sine series**. And Dirichlet's theorem[9] guarantees that virtually *any* function $V_0(y)$—it can even have a finite number of discontinuities—can be expanded in such a series.

But how do we actually *determine* the coefficients C_n, buried as they are in that infinite sum? The device for accomplishing this is so lovely it deserves a name—I call it **Fourier's trick**, though it seems Euler had used essentially the same idea somewhat earlier. Here's how it goes: Multiply Eq. 3.31 by $\sin(n'\pi y/a)$ (where n' is a positive integer), and integrate from 0 to a:

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a)\, \sin(n'\pi y/a)\, dy = \int_0^a V_0(y) \sin(n'\pi y/a)\, dy. \quad (3.32)$$

You can work out the integral on the left for yourself; the answer is

$$\int_0^a \sin(n\pi y/a)\, \sin(n'\pi y/a)\, dy = \begin{cases} 0, & \text{if } n' \neq n, \\ \dfrac{a}{2}, & \text{if } n' = n. \end{cases} \quad (3.33)$$

Thus all the terms in the series drop out, save only the one where $n = n'$, and the left side of Eq. 3.32, reduces to $(a/2)C_{n'}$. *Conclusion:*[10]

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a)\, dy. \quad (3.34)$$

That *does* it: Eq. 3.30 is the solution, with coefficients given by Eq. 3.34. As a concrete example, suppose the strip at $x = 0$ is a metal plate with constant potential V_0 (remember, it's insulated from the grounded plates at $y = 0$ and $y = a$). Then

$$C_n = \frac{2V_0}{a} \int_0^a \sin(n\pi y/a)\, dy = \frac{2V_0}{n\pi}(1 - \cos n\pi) = \begin{cases} 0, & \text{if } n \text{ is even}, \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd}. \end{cases} \quad (3.35)$$

Thus

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a). \quad (3.36)$$

Figure 3.18 is a plot of this potential; Fig. 3.19 shows how the first few terms in the Fourier series combine to make a better and better approximation to the constant V_0: (a) is $n = 1$ only, (b) includes n up to 5, (c) is the sum of the first 10 terms, and (d) is the sum of the first 100 terms.

[9]Boas, M., *Mathematical Methods in the Physical Sciences,* 2nd ed. (New York: John Wiley, 1983).

[10]For aesthetic reasons I've dropped the prime; Eq. 3.34 holds for $n = 1, 2, 3, \ldots$, and it doesn't matter (obviously) what letter you use for the "dummy" index.

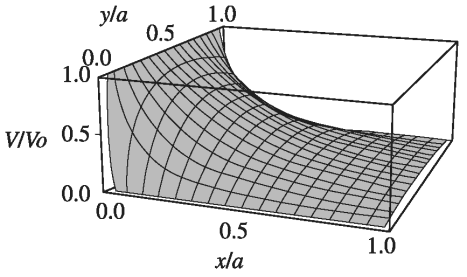


FIGURE 3.18

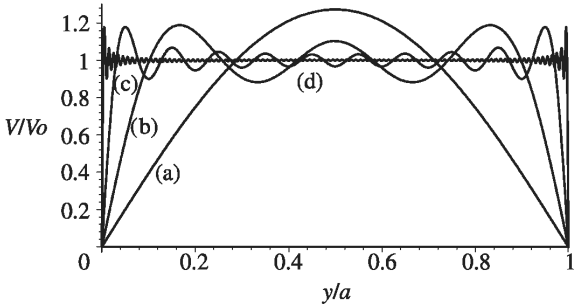


FIGURE 3.19

Incidentally, the infinite series in Eq. 3.36 can be summed explicitly (try your hand at it, if you like); the result is

$$V(x, y) = \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right). \tag{3.37}$$

In this form, it is easy to check that Laplace's equation is obeyed and the four boundary conditions (Eq. 3.21) are satisfied.

The success of this method hinged on two extraordinary properties of the separable solutions (Eqs. 3.28 and 3.29): **completeness** and **orthogonality**. A set of functions $f_n(y)$ is said to be **complete** if any other function $f(y)$ can be expressed as a linear combination of them:

$$f(y) = \sum_{n=1}^{\infty} C_n f_n(y). \tag{3.38}$$

The functions $\sin(n\pi y/a)$ are complete on the interval $0 \leq y \leq a$. It was this fact, guaranteed by Dirichlet's theorem, that assured us Eq. 3.31 could be satisfied, given the proper choice of the coefficients C_n. (The *proof* of completeness, for a particular set of functions, is an extremely difficult business, and I'm afraid

physicists tend to *assume* it's true and leave the checking to others.) A set of functions is **orthogonal** if the integral of the product of any two different members of the set is zero:

$$\int_0^a f_n(y) f_{n'}(y)\, dy = 0 \qquad \text{for } n' \neq n. \tag{3.39}$$

The sine functions are orthogonal (Eq. 3.33); this is the property on which Fourier's trick is based, allowing us to kill off all terms but one in the infinite series and thereby solve for the coefficients C_n. (Proof of orthogonality is generally quite simple, either by direct integration or by analysis of the differential equation from which the functions came.)

Example 3.4. Two infinitely-long grounded metal plates, again at $y = 0$ and $y = a$, are connected at $x = \pm b$ by metal strips maintained at a constant potential V_0, as shown in Fig. 3.20 (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the resulting rectangular pipe.

Solution

Once again, the configuration is independent of z. Our problem is to solve Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0 \text{ when } x = b, \\ \text{(iv)} & V = V_0 \text{ when } x = -b. \end{array}\right\} \tag{3.40}$$

The argument runs as before, up to Eq. 3.27:

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky).$$

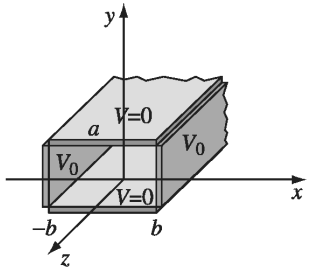


FIGURE 3.20

This time, however, we cannot set $A = 0$; the region in question does not extend to $x = \infty$, so e^{kx} is perfectly acceptable. On the other hand, the situation is *symmetric* with respect to x, so $V(-x, y) = V(x, y)$, and it follows that $A = B$. Using

$$e^{kx} + e^{-kx} = 2\cosh kx,$$

and absorbing $2A$ into C and D, we have

$$V(x, y) = \cosh kx\,(C \sin ky + D \cos ky).$$

Boundary conditions (i) and (ii) require, as before, that $D = 0$ and $k = n\pi/a$, so

$$V(x, y) = C \cosh(n\pi x/a)\, \sin(n\pi y/a). \tag{3.41}$$

Because $V(x, y)$ is even in x, it will automatically meet condition (iv) if it fits (iii). It remains, therefore, to construct the general linear combination,

$$V(x, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a)\, \sin(n\pi y/a),$$

and pick the coefficients C_n in such a way as to satisfy condition (iii):

$$V(b, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi b/a)\, \sin(n\pi y/a) = V_0.$$

This is the same problem in Fourier analysis that we faced before; I quote the result from Eq. 3.35:

$$C_n \cosh(n\pi b/a) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Conclusion: The potential in this case is given by

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \sin(n\pi y/a). \tag{3.42}$$

This function is shown in Fig. 3.21.

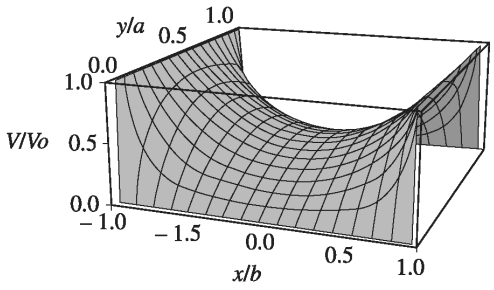


FIGURE 3.21

Example 3.5. An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y, z)$, as indicated in Fig. 3.22. Find the potential inside the pipe.

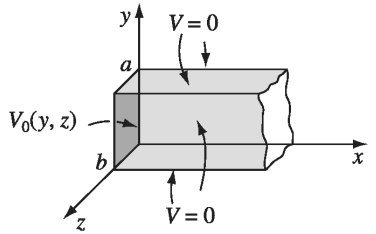


FIGURE 3.22

Solution

This is a genuinely three-dimensional problem,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{3.43}$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = 0 \text{ when } z = 0, \\ \text{(iv)} & V = 0 \text{ when } z = b, \\ \text{(v)} & V \to 0 \text{ as } x \to \infty, \\ \text{(vi)} & V = V_0(y, z) \text{ when } x = 0. \end{array}\right\} \tag{3.44}$$

As always, we look for solutions that are products:

$$V(x, y, z) = X(x)Y(y)Z(z). \tag{3.45}$$

Putting this into Eq. 3.43, and dividing by V, we find

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

It follows that

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \quad \frac{1}{Z}\frac{d^2Z}{dz^2} = C_3, \quad \text{with } C_1 + C_2 + C_3 = 0.$$

Our previous experience (Ex. 3.3) suggests that C_1 must be positive, C_2 and C_3 negative. Setting $C_2 = -k^2$ and $C_3 = -l^2$, we have $C_1 = k^2 + l^2$, and hence

$$\frac{d^2X}{dx^2} = (k^2 + l^2)X, \quad \frac{d^2Y}{dy^2} = -k^2Y, \quad \frac{d^2Z}{dz^2} = -l^2Z. \tag{3.46}$$

Once again, separation of variables has turned a *partial* differential equation into *ordinary* differential equations. The solutions are

$$X(x) = Ae^{\sqrt{k^2+l^2}\,x} + Be^{-\sqrt{k^2+l^2}\,x},$$
$$Y(y) = C\sin ky + D\cos ky,$$
$$Z(z) = E\sin lz + F\cos lz.$$

Boundary condition (v) implies $A = 0$, (i) gives $D = 0$, and (iii) yields $F = 0$, whereas (ii) and (iv) require that $k = n\pi/a$ and $l = m\pi/b$, where n and m are positive integers. Combining the remaining constants, we are left with

$$V(x, y, z) = Ce^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin(n\pi y/a)\, \sin(m\pi z/b). \tag{3.47}$$

This solution meets all the boundary conditions except (vi). It contains *two* unspecified integers (n and m), and the most general linear combination is a *double* sum:

$$V(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin(n\pi y/a)\, \sin(m\pi z/b). \tag{3.48}$$

We hope to fit the remaining boundary condition,

$$V(0, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin(n\pi y/a)\, \sin(m\pi z/b) = V_0(y, z), \tag{3.49}$$

by appropriate choice of the coefficients $C_{n,m}$. To determine these constants, we multiply by $\sin(n'\pi y/a)\,\sin(m'\pi z/b)$, where n' and m' are arbitrary positive integers, and integrate:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\int_0^a \sin(n\pi y/a)\,\sin(n'\pi y/a)\,dy\int_0^b \sin(m\pi z/b)\,\sin(m'\pi z/b)\,dz$$
$$= \int_0^a\int_0^b V_0(y,z)\sin(n'\pi y/a)\,\sin(m'\pi z/b)\,dy\,dz.$$

Quoting Eq. 3.33, the left side is $(ab/4)C_{n',m'}$, so

$$C_{n,m} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin(n\pi y/a)\,\sin(m\pi z/b)\,dy\,dz. \tag{3.50}$$

Equation 3.48, with the coefficients given by Eq. 3.50, is the solution to our problem.

For instance, if the end of the tube is a conductor at *constant* potential V_0,

$$C_{n,m} = \frac{4V_0}{ab}\int_0^a \sin(n\pi y/a)\,dy\int_0^b \sin(m\pi z/b)\,dz$$
$$= \begin{cases} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \dfrac{16V_0}{\pi^2 nm}, & \text{if } n \text{ and } m \text{ are odd.} \end{cases} \tag{3.51}$$

In this case

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n,m=1,3,5\ldots}^{\infty}\frac{1}{nm}\,e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\,\sin(m\pi z/b). \tag{3.52}$$

Notice that the successive terms decrease rapidly; a reasonable approximation would be obtained by keeping only the first few.

Problem 3.13 Find the potential in the infinite slot of Ex. 3.3 if the boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = a/2$, is held at a constant potential V_0, and the other, from $y = a/2$ to $y = a$, is at potential $-V_0$.

Problem 3.14 For the infinite slot (Ex. 3.3), determine the charge density $\sigma(y)$ on the strip at $x = 0$, assuming it is a conductor at constant potential V_0.

Problem 3.15 A rectangular pipe, running parallel to the z-axis (from $-\infty$ to $+\infty$), has three grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specified potential $V_0(y)$.

(a) Develop a general formula for the potential inside the pipe.

(b) Find the potential explicitly, for the case $V_0(y) = V_0$ (a constant).

Problem 3.16 A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (Fig. 3.23). The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box. [What should the potential at the center $(a/2, a/2, a/2)$ be? Check numerically that your formula is consistent with this value.][11]

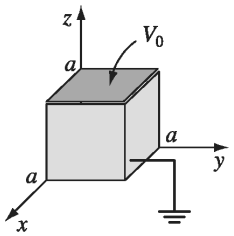


FIGURE 3.23

3.3.2 ■ Spherical Coordinates

In the examples considered so far, Cartesian coordinates were clearly appropriate, since the boundaries were planes. For round objects, spherical coordinates are more natural. In the spherical system, Laplace's equation reads:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0. \tag{3.53}$$

I shall assume the problem has **azimuthal symmetry**, so that V *is independent of* ϕ;[12] in that case, Eq. 3.53 reduces to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0. \tag{3.54}$$

As before, we look for solutions that are products:

$$V(r,\theta)=R(r)\Theta(\theta). \tag{3.55}$$

Putting this into Eq. 3.54, and dividing by V,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0. \tag{3.56}$$

[11]This cute test was suggested by J. Castro.

[12]The general case, for ϕ-dependent potentials, is treated in all the graduate texts. See, for instance, J. D. Jackson's *Classical Electrodynamics*, 3rd ed. (New York: John Wiley, 1999), Chapter 3.

Since the first term depends only on r, and the second only on θ, it follows that each must be a constant:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1), \quad \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1). \tag{3.57}$$

Here $l(l+1)$ is just a fancy way of writing the separation constant—you'll see in a minute why this is convenient.

As always, separation of variables has converted a *partial* differential equation (3.54) into *ordinary* differential equations (3.57). The radial equation,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R, \tag{3.58}$$

has the general solution

$$R(r) = Ar^l + \frac{B}{r^{l+1}}, \tag{3.59}$$

as you can easily check; A and B are the two arbitrary constants to be expected in the solution of a second-order differential equation. But the angular equation,

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta, \tag{3.60}$$

is not so simple. The solutions are **Legendre polynomials** in the variable $\cos\theta$:

$$\Theta(\theta) = P_l(\cos\theta). \tag{3.61}$$

$P_l(x)$ is most conveniently defined by the **Rodrigues formula:**

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l. \tag{3.62}$$

The first few Legendre polynomials are listed in Table 3.1.

$P_0(x)$	=	1
$P_1(x)$	=	x
$P_2(x)$	=	$(3x^2-1)/2$
$P_3(x)$	=	$(5x^3-3x)/2$
$P_4(x)$	=	$(35x^4-30x^2+3)/8$
$P_5(x)$	=	$(63x^5-70x^3+15x)/8$

TABLE 3.1 Legendre Polynomials.

Notice that $P_l(x)$ is (as the name suggests) an lth-order *polynomial* in x; it contains only *even* powers, if l is even, and *odd* powers, if l is odd. The factor in front $(1/2^l l!)$ was chosen in order that

$$P_l(1) = 1. \tag{3.63}$$

The Rodrigues formula obviously works only for nonnegative integer values of l. Moreover, it provides us with only *one* solution. But Eq. 3.60 is *second*-order, and it should possess *two* independent solutions, for *every* value of l. It turns out that these "other solutions" blow up at $\theta = 0$ and/or $\theta = \pi$, and are therefore unacceptable on physical grounds.[13] For instance, the second solution for $l = 0$ is

$$\Theta(\theta) = \ln\left(\tan\frac{\theta}{2}\right). \tag{3.64}$$

You might want to check for yourself that this satisfies Eq. 3.60.

In the case of azimuthal symmetry, then, the most general separable solution to Laplace's equation, consistent with minimal physical requirements, is

$$V(r,\theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right) P_l(\cos\theta).$$

(There was no need to include an overall constant in Eq. 3.61 because it can be absorbed into A and B at this stage.) As before, separation of variables yields an infinite set of solutions, one for each l. The *general* solution is the linear combination of separable solutions:

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta).} \tag{3.65}$$

The following examples illustrate the power of this important result.

Example 3.6. The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R. Find the potential inside the sphere.

Solution

In this case, $B_l = 0$ for all l—otherwise the potential would blow up at the origin. Thus,

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta). \tag{3.66}$$

[13] In rare cases where the z axis is excluded, these "other solutions" do have to be considered.

At $r = R$ this must match the specified function $V_0(\theta)$:

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta). \tag{3.67}$$

Can this equation be satisfied, for an appropriate choice of coefficients A_l? *Yes:* The Legendre polynomials (like the sines) constitute a complete set of functions, on the interval $-1 \le x \le 1$ ($0 \le \theta \le \pi$). How do we determine the constants? Again, by Fourier's trick, for the Legendre polynomials (like the sines) are *orthogonal* functions:[14]

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta$$
$$= \begin{cases} 0, & \text{if } l' \neq l, \\ \dfrac{2}{2l+1}, & \text{if } l' = l. \end{cases} \tag{3.68}$$

Thus, multiplying Eq. 3.67 by $P_{l'}(\cos\theta)\sin\theta$ and integrating, we have

$$A_{l'} R^{l'} \frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta,$$

or

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta\, d\theta. \tag{3.69}$$

Equation 3.66 is the solution to our problem, with the coefficients given by Eq. 3.69.

It can be difficult to evaluate integrals of the form 3.69 analytically, and in practice it is often easier to solve Eq. 3.67 "by eyeball."[15] For instance, suppose we are told that the potential on the sphere is

$$V_0(\theta) = k \sin^2(\theta/2), \tag{3.70}$$

where k is a constant. Using the half-angle formula, we rewrite this as

$$V_0(\theta) = \frac{k}{2}(1 - \cos\theta) = \frac{k}{2}[P_0(\cos\theta) - P_1(\cos\theta)].$$

[14] M. Boas, *Mathematical Methods in the Physical Sciences,* 2nd ed. (New York: John Wiley, 1983), Section 12.7.

[15] This is certainly true whenever $V_0(\theta)$ can be expressed as a polynomial in $\cos\theta$. The degree of the polynomial tells us the highest l we require, and the leading coefficient determines the corresponding A_l. Subtracting off $A_l R^l P_l(\cos\theta)$ and repeating the process, we systematically work our way down to A_0. Notice that if V_0 is an *even* function of $\cos\theta$, then only even terms will occur in the sum (and likewise for odd functions).

Putting this into Eq. 3.67, we read off immediately that $A_0 = k/2$, $A_1 = -k/(2R)$, and all other A_l's vanish. Therefore,

$$V(r,\theta) = \frac{k}{2}\left[r^0 P_0(\cos\theta) - \frac{r^1}{R}P_1(\cos\theta)\right] = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right). \tag{3.71}$$

Example 3.7. The potential $V_0(\theta)$ is again specified on the surface of a sphere of radius R, but this time we are asked to find the potential *outside*, assuming there is no charge there.

Solution

In this case it's the A_l's that must be zero (or else V would not go to zero at ∞), so

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{3.72}$$

At the surface of the sphere, we require that

$$V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta).$$

Multiplying by $P_{l'}(\cos\theta)\sin\theta$ and integrating—exploiting, again, the orthogonality relation 3.68—we have

$$\frac{B_{l'}}{R^{l'+1}}\frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta,$$

or

$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.73}$$

Equation 3.72, with the coefficients given by Eq. 3.73, is the solution to our problem.

Example 3.8. An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. The field will push positive charge to the "northern" surface of the sphere, and—symmetrically—negative charge to the "southern" surface (Fig. 3.24). This induced charge, in turn, distorts the field in the neighborhood of the sphere. Find the potential in the region outside the sphere.

Solution

The sphere is an equipotential—we may as well set it to zero. Then by symmetry the entire xy plane is at potential zero. This time, however, V does *not* go to zero at large z. In fact, far from the sphere the field is $E_0\hat{\mathbf{z}}$, and hence

$$V \to -E_0 z + C.$$

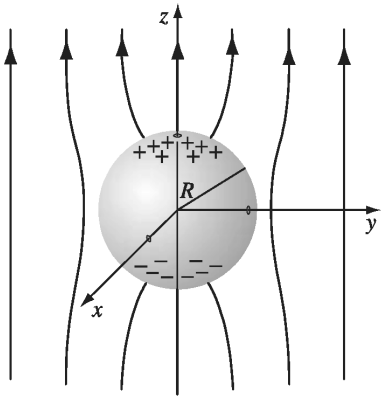


FIGURE 3.24

Since $V = 0$ in the equatorial plane, the constant C must be zero. Accordingly, the boundary conditions for this problem are

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \qquad \text{when } r = R, \\ \text{(ii)} & V \to -E_0 r\cos\theta \qquad \text{for } r \gg R. \end{array}\right\} \tag{3.74}$$

We must fit these boundary conditions with a function of the form 3.65.

The first condition yields

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0,$$

or

$$B_l = -A_l R^{2l+1}, \tag{3.75}$$

so

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta).$$

For $r \gg R$, the second term in parentheses is negligible, and therefore condition (ii) requires that

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta.$$

Evidently only one term is present: $l = 1$. In fact, since $P_1(\cos\theta) = \cos\theta$, we can read off immediately

$$A_1 = -E_0, \quad \text{all other } A_l\text{'s zero.}$$

Conclusion:

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta. \tag{3.76}$$

The first term $(-E_0 r\cos\theta)$ is due to the external field; the contribution attributable to the induced charge is

$$E_0\frac{R^3}{r^2}\cos\theta.$$

If you want to know the induced charge density, it can be calculated in the usual way:

$$\sigma(\theta) = -\epsilon_0\frac{\partial V}{\partial r}\bigg|_{r=R} = \epsilon_0 E_0\left(1 + 2\frac{R^3}{r^3}\right)\cos\theta\bigg|_{r=R} = 3\epsilon_0 E_0\cos\theta. \tag{3.77}$$

As expected, it is positive in the "northern" hemisphere ($0 \le \theta \le \pi/2$) and negative in the "southern" ($\pi/2 \le \theta \le \pi$).

Example 3.9. A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere.

Solution

You could, of course, do this by direct integration:

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma_0}{\mathscr{r}}\,da,$$

but separation of variables is often easier. For the interior region, we have

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r \le R) \tag{3.78}$$

(no B_l terms—they blow up at the origin); in the exterior region

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \qquad (r \ge R) \tag{3.79}$$

(no A_l terms—they don't go to zero at infinity). These two functions must be joined together by the appropriate boundary conditions at the surface itself. First, the potential is *continuous* at $r = R$ (Eq. 2.34):

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta). \tag{3.80}$$

It follows that the coefficients of like Legendre polynomials are equal:

$$B_l = A_l R^{2l+1}. \tag{3.81}$$

(To prove that formally, multiply both sides of Eq. 3.80 by $P_{l'}(\cos\theta)\sin\theta$ and integrate from 0 to π, using the orthogonality relation 3.68.) Second, the radial derivative of V suffers a discontinuity at the surface (Eq. 2.36):

$$\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\Big|_{r=R} = -\frac{1}{\epsilon_0}\sigma_0(\theta). \tag{3.82}$$

Thus

$$-\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_0(\theta),$$

or, using Eq. 3.81,

$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\epsilon_0}\sigma_0(\theta). \tag{3.83}$$

From here, the coefficients can be determined using Fourier's trick:

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.84}$$

Equations 3.78 and 3.79 constitute the solution to our problem, with the coefficients given by Eqs. 3.81 and 3.84.

For instance, if

$$\sigma_0(\theta) = k\cos\theta = kP_1(\cos\theta), \tag{3.85}$$

for some constant k, then all the A_l's are zero except for $l = 1$, and

$$A_1 = \frac{k}{2\epsilon_0}\int_0^{\pi} [P_1(\cos\theta)]^2 \sin\theta\, d\theta = \frac{k}{3\epsilon_0}.$$

The potential inside the sphere is therefore

$$V(r,\theta) = \frac{k}{3\epsilon_0} r\cos\theta \qquad (r \le R), \tag{3.86}$$

whereas outside the sphere

$$V(r,\theta) = \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta \qquad (r \geq R). \tag{3.87}$$

In particular, if $\sigma_0(\theta)$ is the induced charge on a metal sphere in an external field $E_0\hat{\mathbf{z}}$, so that $k = 3\epsilon_0 E_0$ (Eq. 3.77), then the potential inside is $E_0 r\cos\theta = E_0 z$, and the field is $-E_0\hat{\mathbf{z}}$—exactly right to cancel off the external field, as of course it *should* be. Outside the sphere the potential due to this surface charge is

$$E_0\frac{R^3}{r^2}\cos\theta,$$

consistent with our conclusion in Ex. 3.8.

Problem 3.17 Derive $P_3(x)$ from the Rodrigues formula, and check that $P_3(\cos\theta)$ satisfies the angular equation (3.60) for $l = 3$. Check that P_3 and P_1 are orthogonal by explicit integration.

Problem 3.18

(a) Suppose the potential is a *constant* V_0 over the surface of the sphere. Use the results of Ex. 3.6 and Ex. 3.7 to find the potential inside and outside the sphere. (Of course, you know the answers in advance—this is just a consistency check on the method.)

(b) Find the potential inside and outside a spherical shell that carries a uniform surface charge σ_0, using the results of Ex. 3.9.

Problem 3.19 The potential at the surface of a sphere (radius R) is given by

$$V_0 = k\cos 3\theta,$$

where k is a constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. (Assume there's no charge inside or outside the sphere.)

Problem 3.20 Suppose the potential $V_0(\theta)$ at the surface of a sphere is specified, and there is no charge inside or outside the sphere. Show that the charge density on the sphere is given by

$$\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta), \tag{3.88}$$

where

$$C_l = \int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.89}$$

Problem 3.21 Find the potential outside a *charged* metal sphere (charge Q, radius R) placed in an otherwise uniform electric field $\mathbf{E}_0$. Explain clearly where you are setting the zero of potential.

Problem 3.22 In Prob. 2.25, you found the potential on the axis of a uniformly charged disk:

$$V(r,0) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2} - r\right).$$

(a) Use this, together with the fact that $P_l(1) = 1$, to evaluate the first three terms in the expansion (Eq. 3.72) for the potential of the disk at points *off* the axis, assuming $r > R$.

(b) Find the potential for $r < R$ by the same method, using Eq. 3.66. [*Note:* You must break the interior region up into two hemispheres, above and below the disk. Do *not* assume the coefficients A_l are the same in both hemispheres.]

Problem 3.23 A spherical shell of radius R carries a uniform surface charge σ_0 on the "northern" hemisphere and a uniform surface charge $-\sigma_0$ on the "southern" hemisphere. Find the potential inside and outside the sphere, calculating the coefficients explicitly up to A_6 and B_6.

• **Problem 3.24** Solve Laplace's equation by separation of variables in *cylindrical* coordinates, assuming there is no dependence on z (cylindrical symmetry). [Make sure you find *all* solutions to the radial equation; in particular, your result must accommodate the case of an infinite line charge, for which (of course) we already know the answer.]

Problem 3.25 Find the potential outside an infinitely long metal pipe, of radius R, placed at right angles to an otherwise uniform electric field $\mathbf{E}_0$. Find the surface charge induced on the pipe. [Use your result from Prob. 3.24.]

Problem 3.26 Charge density

$$\sigma(\phi) = a \sin 5\phi$$

(where a is a constant) is glued over the surface of an infinite cylinder of radius R (Fig. 3.25). Find the potential inside and outside the cylinder. [Use your result from Prob. 3.24.]

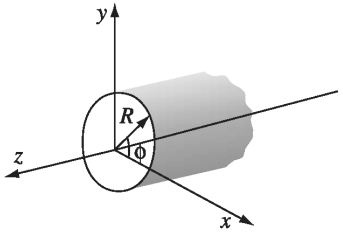


FIGURE 3.25

3.4 ■ MULTIPOLE EXPANSION

3.4.1 ■ Approximate Potentials at Large Distances

If you are very far away from a localized charge distribution, it "looks" like a point charge, and the potential is—to good approximation—$(1/4\pi\epsilon_0)Q/r$, where Q is the total charge. We have often used this as a check on formulas for V. But what if Q is *zero?* You might reply that the potential is then approximately zero, and of course, you're right, in a sense (indeed, the potential at large r is *pretty small* even if Q is *not* zero). But we're looking for something a bit more informative than that.

Example 3.10. A (physical) **electric dipole** consists of two equal and opposite charges ($\pm q$) separated by a distance d. Find the approximate potential at points far from the dipole.

Solution

Let $\mathscr{r}_-$ be the distance from $-q$ and $\mathscr{r}_+$ the distance from $+q$ (Fig. 3.26). Then

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right),$$

and (from the law of cosines)

$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right).$$

We're interested in the régime $r \gg d$, so the third term is negligible, and the binomial expansion yields

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right).$$

Thus

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta,$$

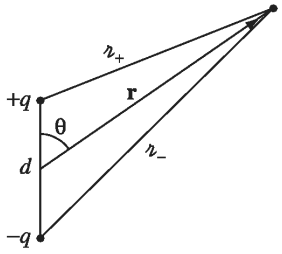


FIGURE 3.26

and hence

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}. \tag{3.90}$$

The potential of a dipole goes like $1/r^2$ at large r; as we might have anticipated, it falls off more rapidly than the potential of a point charge. If we put together a pair of equal and opposite *dipoles* to make a **quadrupole**, the potential goes like $1/r^3$; for back-to-back *quadrupoles* (an **octopole**), it goes like $1/r^4$; and so on. Figure 3.27 summarizes this hierarchy; for completeness I have included the electric **monopole** (point charge), whose potential, of course, goes like $1/r$.

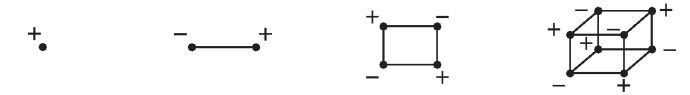


FIGURE 3.27

Example 3.10 pertains to a very special charge configuration. I propose now to develop a systematic expansion for the potential of *any* localized charge distribution, in powers of $1/r$. Figure 3.28 defines the relevant variables; the potential at $\mathbf{r}$ is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\imath}\rho(\mathbf{r}')\,d\tau'. \tag{3.91}$$

Using the law of cosines,

$$\imath^2 = r^2 + (r')^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right],$$

where α is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Thus

$$\imath = r\sqrt{1+\epsilon}, \tag{3.92}$$

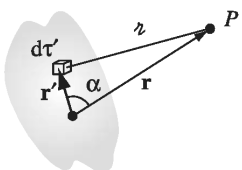


FIGURE 3.28

with

$$\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right).$$

For points well outside the charge distribution, ϵ is much less than 1, and this invites a binomial expansion:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \ldots\right), \tag{3.93}$$

or, in terms of r, r', and α:

$$\begin{aligned}\frac{1}{\mathscr{r}} &= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2\right. \\ &\qquad \left. -\frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \ldots\right] \\ &= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\alpha) + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right)\right. \\ &\qquad \left. + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \ldots\right].\end{aligned}$$

In the last step, I have collected together like powers of (r'/r); surprisingly, their coefficients (the terms in parentheses) are Legendre polynomials! The remarkable result[16] is that

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha). \tag{3.94}$$

Substituting this back into Eq. 3.91, and noting that r is a constant, as far as the integration is concerned, I conclude that

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau',} \tag{3.95}$$

or, more explicitly,

$$\begin{aligned}V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau'\right. \\ \left. + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \ldots\right].\end{aligned} \tag{3.96}$$

[16] This suggests a second way of defining the Legendre polynomials (the first being Rodrigues' formula); $1/\mathscr{r}$ is called the **generating function** for Legendre polynomials.

This is the desired result—the **multipole expansion** of V in powers of $1/r$. The first term ($n = 0$) is the monopole contribution (it goes like $1/r$); the second ($n = 1$) is the dipole (it goes like $1/r^2$); the third is quadrupole; the fourth octopole; and so on. Remember that α is the angle between $\mathbf{r}$ and $\mathbf{r}'$, so the integrals depend on the direction to the field point. If you are interested in the potential along the z' axis (or—putting it the other way around—if you orient your $\mathbf{r}'$ coordinates so the z' axis lies along $\mathbf{r}$), then α is the usual polar angle θ'.

As it stands, Eq. 3.95 is *exact,* but it is *useful* primarily as an approximation scheme: the lowest nonzero term in the expansion provides the approximate potential at large r, and the successive terms tell us how to improve the approximation if greater precision is required.

Problem 3.27 A sphere of radius R, centered at the origin, carries charge density

$$\rho(r, \theta) = k\frac{R}{r^2}(R - 2r)\sin\theta,$$

where k is a constant, and r, θ are the usual spherical coordinates. Find the approximate potential for points on the z axis, far from the sphere.

Problem 3.28 A circular ring in the xy plane (radius R, centered at the origin) carries a uniform line charge λ. Find the first three terms ($n = 0, 1, 2$) in the multipole expansion for $V(r, \theta)$.

3.4.2 ■ The Monopole and Dipole Terms

Ordinarily, the multipole expansion is dominated (at large r) by the monopole term:

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}, \tag{3.97}$$

where $Q = \int \rho\, d\tau$ is the total charge of the configuration. This is just what we expect for the approximate potential at large distances from the charge. For a *point* charge *at the origin,* V_{mon} is the *exact* potential, not merely a first approximation at large r; in this case, all the higher multipoles vanish.

If the total charge is zero, the dominant term in the potential will be the dipole (unless, of course, it *also* vanishes):

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r' \cos\alpha\, \rho(\mathbf{r}')\, d\tau'.$$

Since α is the angle between $\mathbf{r}'$ and $\mathbf{r}$ (Fig. 3.28),

$$r' \cos\alpha = \hat{\mathbf{r}} \cdot \mathbf{r}',$$

and the dipole potential can be written more succinctly:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}} \cdot \int \mathbf{r}'\rho(\mathbf{r}')\, d\tau'.$$

This integral (which does not depend on $\mathbf{r}$) is called the **dipole moment** of the distribution:

$$\boxed{\mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau',} \tag{3.98}$$

and the dipole contribution to the potential simplifies to

$$\boxed{V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}.} \tag{3.99}$$

The dipole moment is determined by the geometry (size, shape, and density) of the charge distribution. Equation 3.98 translates in the usual way (Sect. 2.1.4) for point, line, and surface charges. Thus, the dipole moment of a collection of *point* charges is

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i. \tag{3.100}$$

For a **physical dipole** (equal and opposite charges, $\pm q$),

$$\mathbf{p} = q\mathbf{r}'_+ - q\mathbf{r}'_- = q(\mathbf{r}'_+ - \mathbf{r}'_-) = q\mathbf{d}, \tag{3.101}$$

where $\mathbf{d}$ is the vector from the negative charge to the positive one (Fig. 3.29).

Is this consistent with what we got in Ex. 3.10? Yes: If you put Eq. 3.101 into Eq. 3.99, you recover Eq. 3.90. Notice, however, that this is only the *approximate* potential of the physical dipole—evidently there are higher multipole contributions. Of course, as you go farther and farther away, V_{dip} becomes a better and better approximation, since the higher terms die off more rapidly with increasing r. By the same token, at a fixed r the dipole approximation improves as you shrink the separation d. To construct a **perfect** (point) **dipole** whose potential is given *exactly* by Eq. 3.99, you'd have to let d approach zero. Unfortunately, you then lose the dipole term *too,* unless you simultaneously arrange for q to go to infinity! A *physical* dipole becomes a *pure* dipole, then, in the rather artificial limit $d \to 0$, $q \to \infty$, with the product $qd = p$ held fixed. When someone uses the word "dipole," you can't always tell whether they mean a *physical* dipole (with

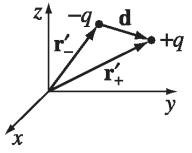


FIGURE 3.29

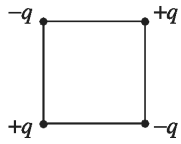


FIGURE 3.30

finite separation between the charges) or an *ideal* (point) dipole. If in doubt, assume that d is small enough (compared to r) that you can safely apply Eq. 3.99.

Dipole moments are *vectors,* and they add accordingly: if you have two dipoles, $\mathbf{p}_1$ and $\mathbf{p}_2$, the total dipole moment is $\mathbf{p}_1 + \mathbf{p}_2$. For instance, with four charges at the corners of a square, as shown in Fig. 3.30, the net dipole moment is zero. You can see this by combining the charges in pairs (vertically, $\downarrow + \uparrow = 0$, or horizontally, $\rightarrow + \leftarrow = 0$) or by adding up the four contributions individually, using Eq. 3.100. This is a *quadrupole,* as I indicated earlier, and its potential is dominated by the quadrupole term in the multipole expansion.

Problem 3.29 Four particles (one of charge q, one of charge $3q$, and two of charge $-2q$) are placed as shown in Fig. 3.31, each a distance a from the origin. Find a simple approximate formula for the potential, valid at points far from the origin. (Express your answer in spherical coordinates.)

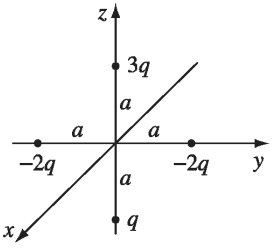


FIGURE 3.31

Problem 3.30 In Ex. 3.9, we derived the exact potential for a spherical shell of radius R, which carries a surface charge $\sigma = k\cos\theta$.

(a) Calculate the dipole moment of this charge distribution.

(b) Find the approximate potential, at points far from the sphere, and compare the exact answer (Eq. 3.87). What can you conclude about the higher multipoles?

Problem 3.31 For the dipole in Ex. 3.10, expand $1/\mathscr{r}_\pm$ to order $(d/r)^3$, and use this to determine the quadrupole and octopole terms in the potential.

3.4.3 ■ Origin of Coordinates in Multipole Expansions

I mentioned earlier that a point charge at the origin constitutes a "pure" monopole. If it is *not* at the origin, it's no longer a pure monopole. For instance, the charge in Fig. 3.32 has a dipole moment $\mathbf{p} = qd\hat{\mathbf{y}}$, and a corresponding dipole term in its potential. The monopole potential $(1/4\pi\epsilon_0)q/r$ is not quite correct for this configuration; rather, the exact potential is $(1/4\pi\epsilon_0)q/\mathscr{r}$. The multipole expansion is, remember, a series in inverse powers of r (the distance to the *origin*), and when we expand $1/\mathscr{r}$, we get *all* powers, not just the first.

So moving the origin (or, what amounts to the same thing, moving the *charge*) can radically alter a multipole expansion. The **monopole moment** Q does not change, since the total charge is obviously independent of the coordinate system. (In Fig. 3.32, the monopole term was unaffected when we moved q away from the origin—it's just that it was no longer the whole story: a dipole term—and for that matter all higher poles—appeared as well.) Ordinarily, the dipole moment *does* change when you shift the origin, but there is an important exception: *If the total charge is zero, then the dipole moment is independent of the choice of origin.* For suppose we displace the origin by an amount $\mathbf{a}$ (Fig. 3.33). The new dipole moment is then

$$\bar{\mathbf{p}} = \int \bar{\mathbf{r}}'\rho(\mathbf{r}')\,d\tau' = \int (\mathbf{r}' - \mathbf{a})\rho(\mathbf{r}')\,d\tau'$$

$$= \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' - \mathbf{a}\int \rho(\mathbf{r}')\,d\tau' = \mathbf{p} - Q\mathbf{a}.$$

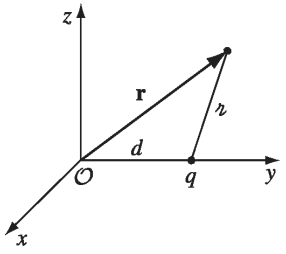


FIGURE 3.32

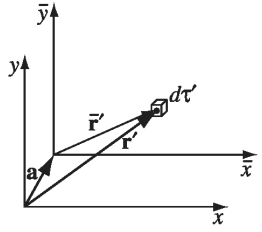
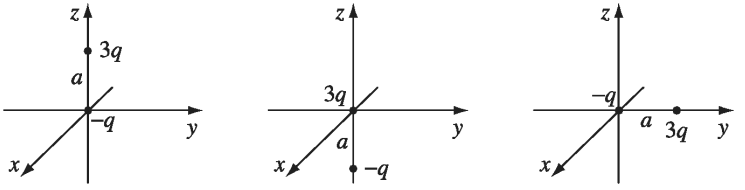


FIGURE 3.33

In particular, if $Q = 0$, then $\bar{\mathbf{p}} = \mathbf{p}$. So if someone asks for the dipole moment in Fig. 3.34(a), you can answer with confidence "$q\mathbf{d}$," but if you're asked for the dipole moment in Fig. 3.34(b), the appropriate response would be "With respect to *what origin*?"

FIGURE 3.34

Problem 3.32 Two point charges, $3q$ and $-q$, are separated by a distance a. For each of the arrangements in Fig. 3.35, find (i) the monopole moment, (ii) the dipole moment, and (iii) the approximate potential (in spherical coordinates) at large r (include both the monopole and dipole contributions).



(a) (b) (c)

FIGURE 3.35

3.4.4 ■ The Electric Field of a Dipole

So far we have worked only with *potentials*. Now I would like to calculate the electric *field* of a (perfect) dipole. If we choose coordinates so that $\mathbf{p}$ is at the origin and points in the z direction (Fig. 3.36), then the potential at r, θ is (Eq. 3.99):

$$V_{\text{dip}}(r,\theta) = \frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}. \tag{3.102}$$

To get the field, we take the negative gradient of V:

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3},$$
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3},$$
$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0.$$

Thus

$$\boxed{\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).} \tag{3.103}$$

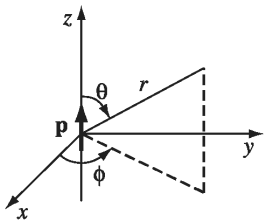
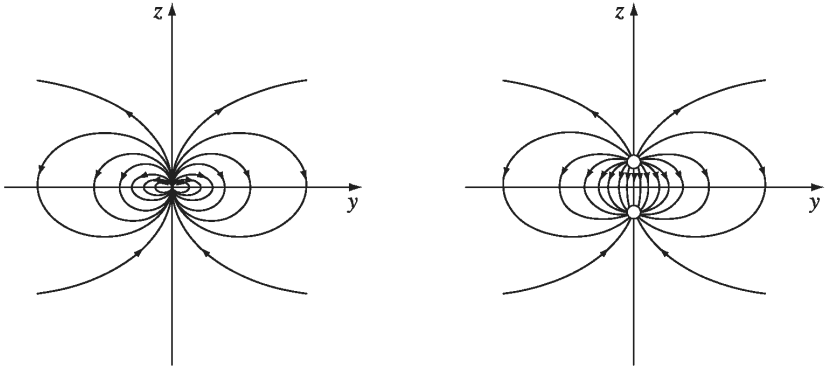


FIGURE 3.36

This formula makes explicit reference to a particular coordinate system (spherical) and assumes a particular orientation for $\mathbf{p}$ (along z). It can be recast in a coordinate-free form, analogous to the potential in Eq. 3.99—see Prob. 3.36.

Notice that the dipole field falls off as the inverse *cube* of r; the *monopole* field $(Q/4\pi\epsilon_0 r^2)\hat{\mathbf{r}}$ goes as the inverse *square,* of course. Quadrupole fields go like $1/r^4$, octopole like $1/r^5$, and so on. (This merely reflects the fact that monopole *potentials* fall off like $1/r$, dipole like $1/r^2$, quadrupole like $1/r^3$, and so on—the gradient introduces another factor of $1/r$.)

Figure 3.37(a) shows the field lines of a "pure" dipole (Eq. 3.103). For comparison, I have also sketched the field lines for a "physical" dipole, in Fig. 3.37(b). Notice how similar the two pictures become if you blot out the central region; up close, however, they are entirely different. Only for points $r \gg d$ does Eq. 3.103 represent a valid approximation to the field of a physical dipole. As I mentioned earlier, this régime can be reached either by going to large r or by squeezing the charges very close together.[17]



(a) Field of a "pure" dipole (b) Field of a "physical" dipole

FIGURE 3.37

[17]Even in the limit, there remains an infinitesimal region at the origin where the field of a physical dipole points in the "wrong" direction, as you can see by "walking" down the z axis in Fig. 3.35(b). If you want to explore this subtle and important point, work Prob. 3.48.

Problem 3.33 A "pure" dipole p is situated at the origin, pointing in the z direction.

(a) What is the force on a point charge q at $(a, 0, 0)$ (Cartesian coordinates)?

(b) What is the force on q at $(0, 0, a)$?

(c) How much work does it take to move q from $(a, 0, 0)$ to $(0, 0, a)$?

Problem 3.34 Three point charges are located as shown in Fig. 3.38, each a distance a from the origin. Find the approximate electric field at points far from the origin. Express your answer in spherical coordinates, and include the two lowest orders in the multipole expansion.

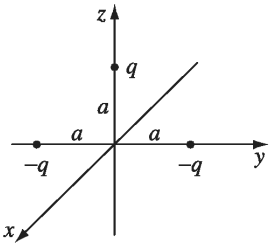


FIGURE 3.38

Problem 3.35 A solid sphere, radius R, is centered at the origin. The "northern" hemisphere carries a uniform charge density ρ_0, and the "southern" hemisphere a uniform charge density $-\rho_0$. Find the approximate field $\mathbf{E}(r, \theta)$ for points far from the sphere ($r \gg R$).

• **Problem 3.36** Show that the electric field of a (perfect) dipole (Eq. 3.103) can be written in the coordinate-free form

$$\boxed{\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}].} \tag{3.104}$$

More Problems on Chapter 3

Problem 3.37 In Section 3.1.4, I proved that the electrostatic potential at any point P in a charge-free region is equal to its average value over any spherical surface (radius R) centered at P. Here's an alternative argument that does not rely on Coulomb's law, only on Laplace's equation. We might as well set the origin at P. Let $V_{\text{ave}}(R)$ be the average; first show that

$$\frac{dV_{\text{ave}}}{dR} = \frac{1}{4\pi R^2}\oint \nabla V \cdot d\mathbf{a}$$

(note that the R^2 in $d\mathbf{a}$ cancels the $1/R^2$ out front, so the only dependence on R is in V itself). Now use the divergence theorem, and conclude that if V satisfies Laplace's equation, then $V_{\text{ave}}(R) = V_{\text{ave}}(0) = V(P)$, for all R.[18]

[18] I thank Ted Jacobson for suggesting this proof.

Problem 3.38 Here's an alternative derivation of Eq. 3.10 (the surface charge density induced on a grounded conducted plane by a point charge q a distance d above the plane). This approach[19] (which generalizes to many other problems) does not rely on the method of images. The total field is due in part to q, and in part to the induced surface charge. Write down the z components of these fields—in terms of q and the as-yet-unknown $\sigma(x, y)$—just below the surface. The sum must be zero, of course, because this is inside a conductor. Use that to determine σ.

Problem 3.39 Two infinite parallel grounded conducting planes are held a distance a apart. A point charge q is placed in the region between them, a distance x from one plate. Find the force on q.[20] Check that your answer is correct for the special cases $a \to \infty$ and $x = a/2$.

Problem 3.40 Two long straight wires, carrying opposite uniform line charges $\pm\lambda$, are situated on either side of a long conducting cylinder (Fig. 3.39). The cylinder (which carries no net charge) has radius R, and the wires are a distance a from the axis. Find the potential.

$$\left[\textit{Answer: } V(s,\phi) = \frac{\lambda}{4\pi\epsilon_0} \ln\left\{\frac{(s^2 + a^2 + 2sa\cos\phi)[(sa/R)^2 + R^2 - 2sa\cos\phi]}{(s^2 + a^2 - 2sa\cos\phi)[(sa/R)^2 + R^2 + 2sa\cos\phi]}\right\}\right]$$

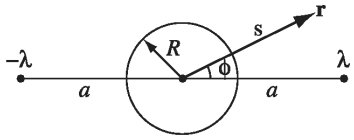


FIGURE 3.39

Problem 3.41 Buckminsterfullerine is a molecule of 60 carbon atoms arranged like the stitching on a soccer-ball. It may be approximated as a conducting spherical shell of radius $R = 3.5$ Å. A nearby electron would be *attracted*, according to Prob. 3.9, so it is not surprising that the ion C_{60}^- exists. (Imagine that the electron—on average—smears itself out uniformly over the surface.) But how about a *second* electron? At large distances it would be *repelled* by the ion, obviously, but at a certain distance r (from the center), the net force is zero, and closer than this it would be attracted. So an electron with enough energy to get in that close should bind.

(a) Find r, in Å. [You'll have to do it numerically.]

(b) How much energy (in electron volts) would it take to push an electron in (from infinity) to the point r?

[Incidentally, the C_{60}^{--} ion has been observed.][21]

[19] See J. L. R. Marrero, *Am. J. Phys.* **78**, 639 (2010).

[20] Obtaining the induced surface charge is not so easy. See B. G. Dick, *Am. J. Phys.* **41**, 1289 (1973), M. Zahn, *Am. J. Phys.* **44**, 1132 (1976), J. Pleines and S. Mahajan, *Am. J. Phys.* **45**, 868 (1977), and Prob. 3.51 below.

[21] Richard Mawhorter suggested this problem.

Problem 3.42 You can use the superposition principle to combine solutions obtained by separation of variables. For example, in Prob. 3.16 you found the potential inside a cubical box, if five faces are grounded and the sixth is at a constant potential V_0; by a six-fold superposition of the result, you could obtain the potential inside a cube with the faces maintained at specified constant voltages V_1, V_2, $\ldots V_6$. In this way, using Ex. 3.4 and Prob. 3.15, find the potential inside a rectangular pipe with two facing sides ($x = \pm b$) at potential V_0, a third ($y = a$) at V_1, and the last (at $y = 0$) grounded.

Problem 3.43 A conducting sphere of radius a, at potential V_0, is surrounded by a thin concentric spherical shell of radius b, over which someone has glued a surface charge

$$\sigma(\theta) = k\cos\theta,$$

where k is a constant and θ is the usual spherical coordinate.

(a) Find the potential in each region: (i) $r > b$, and (ii) $a < r < b$.

(b) Find the induced surface charge $\sigma_i(\theta)$ on the conductor.

(c) What is the total charge of this system? Check that your answer is consistent with the behavior of V at large r.

$$\left[\textit{Answer: } V(r,\theta) = \begin{cases} aV_0/r + (b^3 - a^3)k\cos\theta/3r^2\epsilon_0, & r \geq b \\ aV_0/r + (r^3 - a^3)k\cos\theta/3r^2\epsilon_0, & r \leq b \end{cases}\right]$$

Problem 3.44 A charge $+Q$ is distributed uniformly along the z axis from $z = -a$ to $z = +a$. Show that the electric potential at a point $\mathbf{r}$ is given by

$$V(r,\theta) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\left[1 + \frac{1}{3}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{5}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) + \ldots\right],$$

for $r > a$.

Problem 3.45 A long cylindrical shell of radius R carries a uniform surface charge σ_0 on the upper half and an opposite charge $-\sigma_0$ on the lower half (Fig. 3.40). Find the electric potential inside and outside the cylinder.

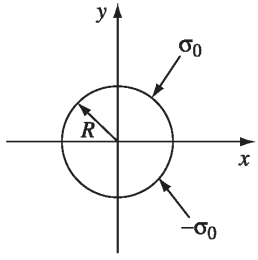


FIGURE 3.40

Problem 3.46 A thin insulating rod, running from $z = -a$ to $z = +a$, carries the indicated line charges. In each case, find the leading term in the multipole expansion of the potential: (a) $\lambda = k\cos(\pi z/2a)$, (b) $\lambda = k\sin(\pi z/a)$, (c) $\lambda = k\cos(\pi z/a)$, where k is a constant.

• **Problem 3.47** Show that the average field inside a sphere of radius R, due to all the charge within the sphere, is

$$\mathbf{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}, \tag{3.105}$$

where $\mathbf{p}$ is the total dipole moment. There are several ways to prove this delightfully simple result. Here's one method:[22]

(a) Show that the average field due to a single charge q at point $\mathbf{r}$ inside the sphere is the same as the field at $\mathbf{r}$ due to a uniformly charged sphere with $\rho = -q/(\frac{4}{3}\pi R^3)$, namely

$$\frac{1}{4\pi\epsilon_0}\frac{1}{(\frac{4}{3}\pi R^3)}\int \frac{q}{\mathcal{r}^2}\hat{\boldsymbol{\mathcal{r}}}\,d\tau',$$

where $\boldsymbol{\mathcal{r}}$ is the vector from $\mathbf{r}$ to $d\tau'$.

(b) The latter can be found from Gauss's law (see Prob. 2.12). Express the answer in terms of the dipole moment of q.

(c) Use the superposition principle to generalize to an arbitrary charge distribution.

(d) While you're at it, show that the average field over the volume of a sphere, due to all the charges *outside*, is the same as the field they produce at the center.

Problem 3.48

(a) Using Eq. 3.103, calculate the average electric field of a dipole, over a spherical volume of radius R, centered at the origin. Do the angular integrals first. [*Note:* You *must* express $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ (see back cover) before integrating. If you don't understand why, reread the discussion in Sect. 1.4.1.] Compare your answer with the general theorem (Eq. 3.105). The discrepancy here is related to the fact that the field of a dipole blows up at $r = 0$. The angular integral is zero, but the radial integral is infinite, so we really don't know *what* to make of the answer. To resolve this dilemma, let's say that Eq. 3.103 applies *outside a tiny sphere of radius* ϵ—its contribution to E_{ave} is then *unambiguously* zero, and the whole answer has to come from the field *inside* the ϵ-sphere.

(b) What must the field *inside* the ϵ-sphere be, in order for the general theorem (Eq. 3.105) to hold? [*Hint:* since ϵ is arbitrarily small, we're talking about something that is infinite at $r = 0$ and whose integral over an infinitesimal volume is finite.] [*Answer:* $-(\mathbf{p}/3\epsilon_0)\delta^3(\mathbf{r})$]

Evidently, the *true* field of a dipole is

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right] - \frac{1}{3\epsilon_0}\mathbf{p}\,\delta^3(\mathbf{r}). \tag{3.106}$$

[22] Another method exploits the result of Prob. 3.4. See B. Y.-K. Hu, *Eur. J. Phys.* **30**, L29 (2009).

You may wonder how we missed the delta-function term[23] when we calculated the field back in Sect. 3.4.4. The answer is that the differentiation leading to Eq. 3.103 is valid *except* at $r = 0$, but we should have known (from our experience in Sect. 1.5.1) that the point $r = 0$ would be problematic.[24]

Problem 3.49 In Ex. 3.9, we obtained the potential of a spherical shell with surface charge $\sigma(\theta) = k\cos\theta$. In Prob. 3.30, you found that the field is pure dipole outside; it's *uniform* inside (Eq. 3.86). Show that the limit $R \to 0$ reproduces the delta function term in Eq. 3.106.

Problem 3.50

(a) Suppose a charge distribution $\rho_1(\mathbf{r})$ produces a potential $V_1(\mathbf{r})$, and some other charge distribution $\rho_2(\mathbf{r})$ produces a potential $V_2(\mathbf{r})$. [The two situations may have nothing in common, for all I care—perhaps number 1 is a uniformly charged sphere and number 2 is a parallel-plate capacitor. Please understand that ρ_1 and ρ_2 are not present *at the same time;* we are talking about two *different problems,* one in which only ρ_1 is present, and another in which only ρ_2 is present.] Prove **Green's reciprocity theorem:**[25]

$$\int_{\text{all space}} \rho_1 V_2 \, d\tau = \int_{\text{all space}} \rho_2 V_1 \, d\tau.$$

[*Hint:* Evaluate $\int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d\tau$ two ways, first writing $\mathbf{E}_1 = -\nabla V_1$ and using integration by parts to transfer the derivative to $\mathbf{E}_2$, then writing $\mathbf{E}_2 = -\nabla V_2$ and transferring the derivative to $\mathbf{E}_1$.]

(b) Suppose now that you have two separated conductors (Fig. 3.41). If you charge up conductor a by amount Q (leaving b uncharged), the resulting potential of b is, say, V_{ab}. On the other hand, if you put that same charge Q on conductor b (leaving a uncharged), the potential of a would be V_{ba}. Use Green's reciprocity theorem to show that $V_{ab} = V_{ba}$ (an astonishing result, since we assumed nothing about the shapes or placement of the conductors).

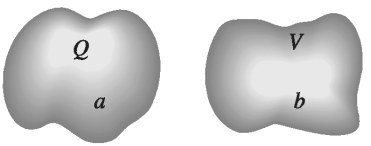


FIGURE 3.41

[23]There are other ways of getting the delta-function term in the field of a dipole—my own favorite is Prob. 3.49. Note that unless you are right on *top* of the dipole, Eq. 3.104 is perfectly adequate.

[24]See C. P. Frahm, *Am. J. Phys.* **51**, 826 (1983). For applications, see D. J. Griffiths, *Am. J. Phys.* **50,** 698 (1982). There are other (perhaps preferable) ways of expressing the **contact** (delta-function) **term** in Eq. 3.106; see A. Gsponer, *Eur. J. Phys.* **28**, 267 (2007), J. Franklin, *Am. J. Phys.* **78**, 1225 (2010), and V. Hnizdo, *Eur. J. Phys.* **32**, 287 (2011).

[25]For interesting commentary, see B. Y.-K. Hu, *Am. J. Phys.* **69**, 1280 (2001).

Problem 3.51 Use Green's reciprocity theorem (Prob. 3.50) to solve the following two problems. [*Hint:* for distribution 1, use the actual situation; for distribution 2, remove q, and set one of the conductors at potential V_0.]

(a) Both plates of a parallel-plate capacitor are grounded, and a point charge q is placed between them at a distance x from plate 1. The plate separation is d. Find the induced charge on each plate. [*Answer:* $Q_1 = q(x/d - 1)$; $Q_2 = -qx/d$]

(b) Two concentric spherical conducting shells (radii a and b) are grounded, and a point charge q is placed between them (at radius r). Find the induced charge on each sphere.

Problem 3.52

(a) Show that the quadrupole term in the multipole expansion can be written

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j Q_{ij}$$

(in the notation of Eq. 1.31), where

$$Q_{ij} \equiv \frac{1}{2}\int [3r'_i r'_j - (r')^2\delta_{ij}]\rho(\mathbf{r}')\,d\tau'.$$

Here

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

is the **Kronecker delta**, and Q_{ij} is the **quadrupole moment** of the charge distribution. Notice the hierarchy:

$$V_{\text{mon}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}; \quad V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\sum \hat{r}_i p_i}{r^2}; \quad V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{\sum \hat{r}_i\hat{r}_j Q_{ij}}{r^3}; \ \ldots$$

The monopole moment (Q) is a scalar, the dipole moment ($\mathbf{p}$) is a vector, the quadrupole moment (Q_{ij}) is a second-rank tensor, and so on.

(b) Find all nine components of Q_{ij} for the configuration in Fig. 3.30 (assume the square has side a and lies in the xy plane, centered at the origin).

(c) Show that the quadrupole moment is independent of origin if the monopole and dipole moments both vanish. (This works all the way up the hierarchy—the lowest nonzero multipole moment is always independent of origin.)

(d) How would you define the **octopole moment**? Express the octopole term in the multipole expansion in terms of the octopole moment.

Problem 3.53 In Ex. 3.8 we determined the electric field outside a spherical conductor (radius R) placed in a uniform external field $\mathbf{E}_0$. Solve the problem now using the method of images, and check that your answer agrees with Eq. 3.76. [*Hint:* Use Ex. 3.2, but put another charge, $-q$, diametrically opposite q. Let $a \to \infty$, with $(1/4\pi\epsilon_0)(2q/a^2) = -E_0$ held constant.]

! **Problem 3.54** For the infinite rectangular pipe in Ex. 3.4, suppose the potential on the bottom ($y = 0$) and the two sides ($x = \pm b$) is zero, but the potential on the top ($y = a$) is a nonzero constant V_0. Find the potential inside the pipe. [*Note:* This is a rotated version of Prob. 3.15(b), but set it up as in Ex. 3.4, using sinusoidal functions in y and hyperbolics in x. It is an unusual case in which $k = 0$ must be included. Begin by finding the general solution to Eq. 3.26 when $k = 0$.][26]

$\left[\textit{Answer: } V_0 \left(\frac{y}{a} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \sin(n\pi y/a) \right)\right.$. Alternatively, using sinusoidal functions of x and hyperbolics in y, $-\frac{2V_0}{b} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh(\alpha_n y)}{\alpha_n \sinh(\alpha_n a)} \cos(\alpha_n x)$, where $\left.\alpha_n \equiv (2n-1)\pi/2b\right]$

! **Problem 3.55**

(a) A long metal pipe of square cross-section (side a) is grounded on three sides, while the fourth (which is insulated from the rest) is maintained at constant potential V_0. Find the net charge per unit length on the side *opposite* to V_0. [*Hint:* Use your answer to Prob. 3.15 or Prob. 3.54.]

(b) A long metal pipe of circular cross-section (radius R) is divided (lengthwise) into four equal sections, three of them grounded and the fourth maintained at constant potential V_0. Find the net charge per unit length on the section opposite to V_0. [*Answer to both (a) and (b):* $\lambda = -(\epsilon_0 V_0/\pi) \ln 2$][27]

Problem 3.56 An ideal electric dipole is situated at the origin, and points in the z direction, as in Fig. 3.36. An electric charge is released from rest at a point in the xy plane. Show that it swings back and forth in a semi-circular arc, as though it were a pendulum supported at the origin.[28]

Problem 3.57 A stationary electric dipole $\mathbf{p} = p\,\hat{z}$ is situated at the origin. A positive point charge q (mass m) executes circular motion (radius s) at constant speed in the field of the dipole. Characterize the plane of the orbit. Find the speed, angular momentum and total energy of the charge.[29] $\left[\textit{Answer: } L = \sqrt{qpm/3\sqrt{3}\pi\epsilon_0}\right]$

Problem 3.58 Find the charge density $\sigma(\theta)$ on the surface of a sphere (radius R) that produces the same electric field, for points exterior to the sphere, as a charge q at the point $a < R$ on the z axis. $\left[\textit{Answer: } \frac{q}{4\pi R}(R^2 - a^2)(R^2 + a^2 - 2Ra\cos\theta)^{-3/2}\right]$

[26] For further discussion, see S. Hassani, *Am. J. Phys.* **59**, 470 (1991).

[27] These are special cases of the **Thompson-Lampard theorem**; see J. D. Jackson, *Am. J. Phys.* **67**, 107 (1999).

[28] This charming result is due to R. S. Jones, *Am. J. Phys.* **63**, 1042 (1995).

[29] G. P. Sastry, V. Srinivas, and A. V. Madhav, *Eur. J. Phys.* **17**, 275 (1996).

CHAPTER

4 Electric Fields in Matter

4.1 ■ POLARIZATION

4.1.1 ■ Dielectrics

In this chapter, we shall study electric fields in matter. Matter, of course, comes in many varieties—solids, liquids, gases, metals, woods, glasses—and these substances do not all respond in the same way to electrostatic fields. Nevertheless, *most* everyday objects belong (at least, in good approximation) to one of two large classes: **conductors** and **insulators** (or **dielectrics**). We have already talked about conductors; these are substances that contain an "unlimited" supply of charges that are free to move about through the material. In practice, what this ordinarily means is that many of the electrons (one or two per atom, in a typical metal) are not associated with any particular nucleus, but roam around at will. In dielectrics, by contrast, *all charges are attached to specific atoms or molecules*—they're on a tight leash, and all they can do is move a bit *within* the atom or molecule. Such microscopic displacements are not as dramatic as the wholesale rearrangement of charge in a conductor, but their cumulative effects account for the characteristic behavior of dielectric materials. There are actually *two* principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule: *stretching* and *rotating*. In the next two sections I'll discuss these processes.

4.1.2 ■ Induced Dipoles

What happens to a neutral atom when it is placed in an electric field $\mathbf{E}$? Your first guess might well be: "Absolutely nothing—since the atom is not charged, the field has no effect on it." But that is incorrect. Although the atom as a whole is electrically neutral, there *is* a positively charged core (the nucleus) and a negatively charged electron cloud surrounding it. These two regions of charge within the atom are influenced by the field: the nucleus is pushed in the direction of the field, and the electrons the opposite way. In principle, if the field is large enough, it can pull the atom apart completely, "ionizing" it (the substance then becomes a conductor). With less extreme fields, however, an equilibrium is soon established, for if the center of the electron cloud does not coincide with the nucleus, these positive and negative charges attract one another, and that holds the atom together. The two opposing forces—$\mathbf{E}$ pulling the electrons and nucleus apart, their mutual attraction drawing them back together—reach a balance, leaving the

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.67	0.396	24.1	1.64	43.4	59.4

TABLE 4.1 Atomic Polarizabilities ($\alpha/4\pi\epsilon_0$, in units of 10^{-30} m^3). *Data from: Handbook of Chemistry and Physics,* 91st ed. (Boca Raton: CRC Press, 2010).

atom **polarized,** with plus charge shifted slightly one way, and minus the other. The atom now has a tiny dipole moment **p**, which points in the *same direction as* **E**. Typically, this induced dipole moment is approximately proportional to the field (as long as the latter is not too strong):

$$\mathbf{p} = \alpha \mathbf{E}. \tag{4.1}$$

The constant of proportionality α is called **atomic polarizability.** Its value depends on the detailed structure of the atom in question. Table 4.1 lists some experimentally determined atomic polarizabilities.

Example 4.1. A primitive model for an atom consists of a point nucleus $(+q)$ surrounded by a uniformly charged spherical cloud $(-q)$ of radius a (Fig. 4.1). Calculate the atomic polarizability of such an atom.

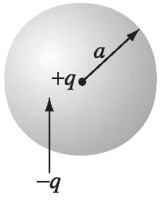


FIGURE 4.1

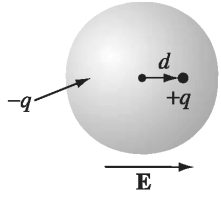


FIGURE 4.2

Solution

In the presence of an external field **E**, the nucleus will be shifted slightly to the right and the electron cloud to the left, as shown in Fig. 4.2. (Because the actual displacements involved are extremely small, as you'll see in Prob. 4.1, it is reasonable to assume that the electron cloud retains its spherical shape.) Say that equilibrium occurs when the nucleus is displaced a distance d from the center of the sphere. At that point, the external field pushing the nucleus to the right exactly balances the internal field pulling it to the left: $E = E_e$, where E_e is the field produced by the electron cloud. Now the field at a distance d from the center of a uniformly charged sphere is

$$E_e = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}$$

(Prob. 2.12). At equilibrium, then,

$$E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}, \quad \text{or} \quad p = qd = (4\pi\epsilon_0 a^3)E.$$

The atomic polarizability is therefore

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v, \tag{4.2}$$

where v is the volume of the atom. Although this atomic model is extremely crude, the result (Eq. 4.2) is not too bad—it's accurate to within a factor of four or so for many simple atoms.

For molecules the situation is not quite so simple, because frequently they polarize more readily in some directions than in others. Carbon dioxide (Fig. 4.3), for instance, has a polarizability of 4.5×10^{-40} C^2·m/N when you apply the field along the axis of the molecule, but only 2×10^{-40} for fields perpendicular to this direction. When the field is at some *angle* to the axis, you must resolve it into parallel and perpendicular components, and multiply each by the pertinent polarizability:

$$\mathbf{p} = \alpha_\perp \mathbf{E}_\perp + \alpha_\parallel \mathbf{E}_\parallel.$$

In this case, the induced dipole moment may not even be in the same *direction* as $\mathbf{E}$. And CO_2 is relatively simple, as molecules go, since the atoms at least arrange themselves in a straight line; for a completely asymmetrical molecule, Eq. 4.1 is replaced by the most general linear relation between $\mathbf{E}$ and $\mathbf{p}$:

$$\left.\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{aligned}\right\} \tag{4.3}$$

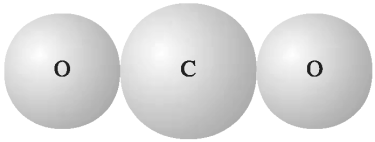


FIGURE 4.3

The set of nine constants α_{ij} constitute the **polarizability tensor** for the molecule. Their values depend on the orientation of the axes you use, though it is always possible to choose "principal" axes such that all the off-diagonal terms (α_{xy}, α_{zx}, etc.) vanish, leaving just three nonzero polarizabilities: α_{xx}, α_{yy}, and α_{zz}.

Problem 4.1 A hydrogen atom (with the Bohr radius of half an angstrom) is situated between two metal plates 1 mm apart, which are connected to opposite terminals of a 500 V battery. What fraction of the atomic radius does the separation distance d amount to, roughly? Estimate the voltage you would need with this apparatus to ionize the atom. [Use the value of α in Table 4.1. *Moral:* The displacements we're talking about are *minute,* even on an atomic scale.]

Problem 4.2 According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom. [*Hint:* First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assuming $r \ll a$.[1]

Problem 4.3 According to Eq. 4.1, the induced dipole moment of an atom is proportional to the external field. This is a "rule of thumb," not a fundamental law, and it is easy to concoct exceptions—in theory. Suppose, for example, the charge density of the electron cloud were proportional to the distance from the center, out to a radius R. To what power of E would p be proportional in that case? Find the condition on $\rho(r)$ such that Eq. 4.1 will hold in the weak-field limit.

Problem 4.4 A point charge q is situated a large distance r from a neutral atom of polarizability α. Find the force of attraction between them.

4.1.3 ■ Alignment of Polar Molecules

The neutral atom discussed in Sect. 4.1.2 had no dipole moment to start with—$\mathbf{p}$ was *induced* by the applied field. Some molecules have built-in, permanent dipole moments. In the water molecule, for example, the electrons tend to cluster around the oxygen atom (Fig. 4.4), and since the molecule is bent at 105°, this leaves a negative charge at the vertex and a net positive charge on the opposite side. (The dipole moment of water is unusually large: 6.1×10^{-30} C·m; in fact, this is what accounts for its effectiveness as a solvent.) What happens when such molecules (called **polar molecules**) are placed in an electric field?

If the field is uniform, the *force* on the positive end, $\mathbf{F}_+ = q\mathbf{E}$, exactly cancels the force on the negative end, $\mathbf{F}_- = -q\mathbf{E}$ (Fig. 4.5). However, there will be a *torque:*

$$\begin{aligned}\mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\ &= \left[(\mathbf{d}/2) \times (q\mathbf{E})\right] + \left[(-\mathbf{d}/2) \times (-q\mathbf{E})\right] = q\mathbf{d} \times \mathbf{E}.\end{aligned}$$

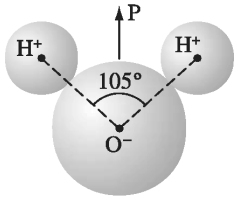


FIGURE 4.4

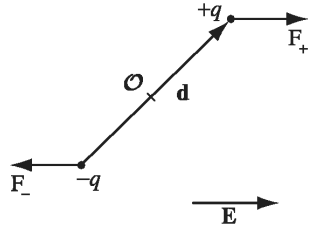


FIGURE 4.5

[1]For a more sophisticated approach, see W. A. Bowers, *Am. J. Phys.* **54,** 347 (1986).

Thus a dipole $\mathbf{p} = q\mathbf{d}$ in a uniform field $\mathbf{E}$ experiences a torque

$$\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}.} \tag{4.4}$$

Notice that $\mathbf{N}$ is in such a direction as to line $\mathbf{p}$ up *parallel* to $\mathbf{E}$; a polar molecule that is free to rotate will swing around until it points in the direction of the applied field.

If the field is *non*uniform, so that $\mathbf{F}_+$ does not exactly balance $\mathbf{F}_-$, there will be a net *force* on the dipole, in addition to the torque. Of course, $\mathbf{E}$ must change rather abruptly for there to be significant variation in the space of one molecule, so this is not ordinarily a major consideration in discussing the behavior of dielectrics. Nevertheless, the formula for the force on a dipole in a nonuniform field is of some interest:

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ - \mathbf{E}_-) = q(\Delta\mathbf{E}),$$

where $\Delta\mathbf{E}$ represents the difference between the field at the plus end and the field at the minus end. Assuming the dipole is very short, we may use Eq. 1.35 to approximate the small change in E_x:

$$\Delta E_x \equiv (\nabla E_x) \cdot \mathbf{d},$$

with corresponding formulas for E_y and E_z. More compactly,

$$\Delta\mathbf{E} = (\mathbf{d} \cdot \nabla)\mathbf{E},$$

and therefore[2]

$$\boxed{\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}.} \tag{4.5}$$

For a "perfect" dipole of infinitesimal length, Eq. 4.4 gives the torque *about the center of the dipole* even in a *non*uniform field; about any *other* point $\mathbf{N} = (\mathbf{p} \times \mathbf{E}) + (\mathbf{r} \times \mathbf{F})$.

Problem 4.5 In Fig. 4.6, $\mathbf{p}_1$ and $\mathbf{p}_2$ are (perfect) dipoles a distance r apart. What is the torque on $\mathbf{p}_1$ due to $\mathbf{p}_2$? What is the torque on $\mathbf{p}_2$ due to $\mathbf{p}_1$? [In each case, I want the torque on the dipole *about its own center*. If it bothers you that the answers are not equal and opposite, see Prob. 4.29.]

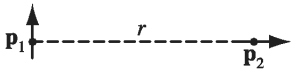


FIGURE 4.6

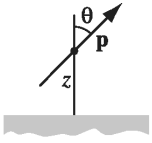


FIGURE 4.7

[2]In the present context, Eq. 4.5 could be written more conveniently as $\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E})$. However, it is safer to stick with $(\mathbf{p} \cdot \nabla)\mathbf{E}$, because we will be applying the formula to materials in which the dipole moment (per unit volume) is itself a function of position and this second expression would imply (incorrectly) that $\mathbf{p}$ *too* is to be differentiated.

Problem 4.6 A (perfect) dipole $\mathbf{p}$ is situated a distance z above an infinite grounded conducting plane (Fig. 4.7). The dipole makes an angle θ with the perpendicular to the plane. Find the torque on $\mathbf{p}$. If the dipole is free to rotate, in what orientation will it come to rest?

Problem 4.7 Show that the energy of an ideal dipole $\mathbf{p}$ in an electric field $\mathbf{E}$ is given by

$$\boxed{U = -\mathbf{p} \cdot \mathbf{E}.} \tag{4.6}$$

Problem 4.8 Show that the interaction energy of two dipoles separated by a displacement $\mathbf{r}$ is

$$U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})(\mathbf{p}_2 \cdot \hat{\mathbf{r}})]. \tag{4.7}$$

[*Hint:* Use Prob. 4.7 and Eq. 3.104.]

Problem 4.9 A dipole $\mathbf{p}$ is a distance r from a point charge q, and oriented so that $\mathbf{p}$ makes an angle θ with the vector $\mathbf{r}$ from q to $\mathbf{p}$.

(a) What is the force on $\mathbf{p}$?

(b) What is the force on q?

4.1.4 ■ Polarization

In the previous two sections, we have considered the effect of an external electric field on an individual atom or molecule. We are now in a position to answer (qualitatively) the original question: What happens to a piece of dielectric material when it is placed in an electric field? If the substance consists of neutral atoms (or nonpolar molecules), the field will induce in each a tiny dipole moment, pointing in the same direction as the field.[3] If the material is made up of polar molecules, each permanent dipole will experience a torque, tending to line it up along the field direction. (Random thermal motions compete with this process, so the alignment is never complete, especially at higher temperatures, and disappears almost at once when the field is removed.)

Notice that these two mechanisms produce the same basic result: *a lot of little dipoles pointing along the direction of the field*—the material becomes **polarized**. A convenient measure of this effect is

$$\mathbf{P} \equiv \textit{dipole moment per unit volume,}$$

which is called the **polarization**. From now on we shall not worry much about how the polarization *got* there. Actually, the two mechanisms I described are not as clear-cut as I tried to pretend. Even in polar molecules there will be

[3]In asymmetric molecules, the induced dipole moment may not be parallel to the field, but if the molecules are randomly oriented, the perpendicular contributions will *average* to zero. Within a single crystal, the orientations are certainly *not* random, and we would have to treat this case separately.

some polarization by displacement (though generally it is a lot easier to rotate a molecule than to stretch it, so the second mechanism dominates). It's even possible in some materials to "freeze in" polarization, so that it persists after the field is removed. But let's forget for a moment about the *cause* of the polarization, and let's study the field that a chunk of polarized material *itself* produces. Then, in Sect. 4.3, we'll put it all together: the original field, which was *responsible* for **P**, plus the new field, which is *due* to **P**.

4.2 ■ THE FIELD OF A POLARIZED OBJECT

4.2.1 ■ Bound Charges

Suppose we have a piece of polarized material—that is, an object containing a lot of microscopic dipoles lined up. The dipole moment per unit volume **P** is given. *Question:* What is the field produced by this object (not the field that may have *caused* the polarization, but the field the polarization *itself* causes)? Well, we know what the field of an individual dipole looks like, so why not chop the material up into infinitesimal dipoles and integrate to get the total? As usual, it's easier to work with the potential. For a single dipole **p** (Eq. 3.99),

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\boldsymbol{\imath}}}{\imath^2}, \tag{4.8}$$

where $\boldsymbol{\imath}$ is the vector from the dipole to the point at which we are evaluating the potential (Fig. 4.8). In the present context, we have a dipole moment $\mathbf{p} = \mathbf{P}\,d\tau'$ in each volume element $d\tau'$, so the total potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau'. \tag{4.9}$$

That *does* it, in principle. But a little sleight-of-hand casts this integral into a much more illuminating form. Observing that

$$\boldsymbol{\nabla}'\left(\frac{1}{\imath}\right) = \frac{\hat{\boldsymbol{\imath}}}{\imath^2},$$

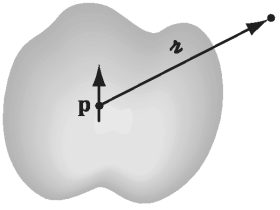


FIGURE 4.8

where (unlike Prob. 1.13) the differentiation is with respect to the *source* coordinates ($\mathbf{r}'$), we have

$$V = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \mathbf{P}\cdot\boldsymbol{\nabla}'\left(\frac{1}{\imath}\right)\,d\tau'.$$

Integrating by parts, using product rule number 5 (in the front cover), gives

$$V = \frac{1}{4\pi\epsilon_0}\left[\int_{\mathcal{V}} \boldsymbol{\nabla}'\cdot\left(\frac{\mathbf{P}}{\imath}\right)\,d\tau' - \int_{\mathcal{V}} \frac{1}{\imath}(\boldsymbol{\nabla}'\cdot\mathbf{P})\,d\tau'\right],$$

or, invoking the divergence theorem,

$$V = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{1}{\imath}\mathbf{P}\cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{1}{\imath}(\boldsymbol{\nabla}'\cdot\mathbf{P})\,d\tau'. \tag{4.10}$$

The first term looks like the potential of a surface charge

$$\boxed{\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}} \tag{4.11}$$

(where $\hat{\mathbf{n}}$ is the normal unit vector), while the second term looks like the potential of a volume charge

$$\boxed{\rho_b \equiv -\boldsymbol{\nabla}\cdot\mathbf{P}.} \tag{4.12}$$

With these definitions, Eq. 4.10 becomes

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b}{\imath}\,da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b}{\imath}\,d\tau'. \tag{4.13}$$

What this means is that the potential (and hence also the field) of a polarized object is the same as that produced by a volume charge density $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P}$ plus a surface charge density $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$. Instead of integrating the contributions of all the infinitesimal dipoles, as in Eq. 4.9, we could first find those **bound charges**, and then calculate the fields *they* produce, in the same way we calculate the field of any other volume and surface charges (for example, using Gauss's law).

Example 4.2. Find the electric field produced by a uniformly polarized sphere of radius R.

Solution

We may as well choose the z axis to coincide with the direction of polarization (Fig. 4.9). The volume bound charge density ρ_b is zero, since $\mathbf{P}$ is uniform, but

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P\cos\theta,$$

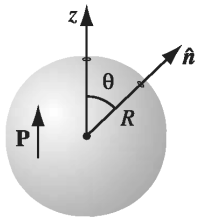


FIGURE 4.9

where θ is the usual spherical coordinate. What we want, then, is the field produced by a charge density $P\cos\theta$ plastered over the surface of a sphere. But we already computed the potential of such a configuration, in Ex. 3.9:

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta, & \text{for} \quad r \le R, \\ \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for} \quad r \ge R. \end{cases}$$

Since $r\cos\theta = z$, the *field* inside the sphere is *uniform*:

$$\mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{1}{3\epsilon_0}\mathbf{P}, \quad \text{for} \quad r < R. \tag{4.14}$$

This remarkable result will be very useful in what follows. Outside the sphere the potential is identical to that of a perfect dipole at the origin,

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \quad \text{for} \quad r \ge R, \tag{4.15}$$

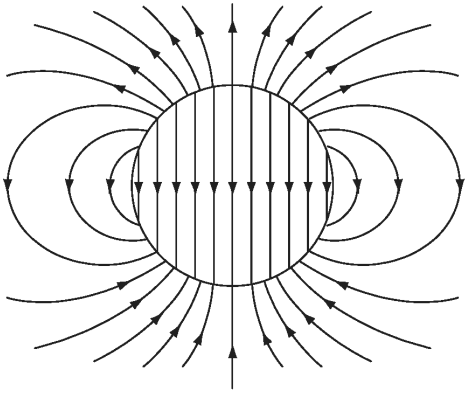


FIGURE 4.10

whose dipole moment is, not surprisingly, equal to the total dipole moment of the sphere:

$$\mathbf{p} = \frac{4}{3}\pi R^3 \mathbf{P}. \tag{4.16}$$

The field of the uniformly polarized sphere is shown in Fig. 4.10.

Problem 4.10 A sphere of radius R carries a polarization

$$\mathbf{P}(\mathbf{r}) = k\mathbf{r},$$

where k is a constant and $\mathbf{r}$ is the vector from the center.

(a) Calculate the bound charges σ_b and ρ_b.

(b) Find the field inside and outside the sphere.

Problem 4.11 A short cylinder, of radius a and length L, carries a "frozen-in" uniform polarization $\mathbf{P}$, parallel to its axis. Find the bound charge, and sketch the electric field (i) for $L \gg a$, (ii) for $L \ll a$, and (iii) for $L \approx a$. [This is known as a **bar electret**; it is the electrical analog to a bar magnet. In practice, only very special materials—barium titanate is the most "familiar" example—will hold a permanent electric polarization. That's why you can't buy electrets at the toy store.]

Problem 4.12 Calculate the potential of a uniformly polarized sphere (Ex. 4.2) directly from Eq. 4.9.

4.2.2 ■ Physical Interpretation of Bound Charges

In the last section we found that the field of a polarized object is identical to the field that would be produced by a certain distribution of "bound charges," σ_b and ρ_b. But this conclusion emerged in the course of abstract manipulations on the integral in Eq. 4.9, and left us with no clue as to the physical meaning of these bound charges. Indeed, some authors give you the impression that bound charges are in some sense "fictitious"—mere bookkeeping devices used to facilitate the calculation of fields. Nothing could be further from the truth: ρ_b and σ_b represent *perfectly genuine accumulations of charge.* In this section I'll explain how polarization leads to these charge distributions.

The basic idea is very simple: Suppose we have a long string of dipoles, as shown in Fig. 4.11. Along the line, the head of one effectively cancels the tail of its neighbor, but at the ends there are two charges left over: plus at the right end and minus at the left. It is as if we had peeled off an electron at one end and carried it all the way down to the other end, though in fact no single electron made the whole trip—a lot of tiny displacements add up to one large one. We call the net charge at the ends a *bound* charge to remind ourselves that it cannot be removed;

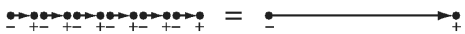


FIGURE 4.11

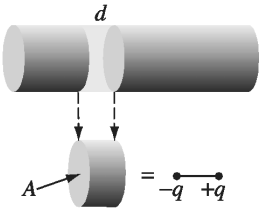


FIGURE 4.12

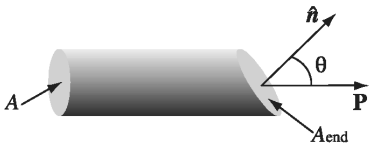


FIGURE 4.13

in a dielectric every electron is attached to a specific atom or molecule. But apart from that, bound charge is no different from any other kind.

To calculate the actual *amount* of bound charge resulting from a given polarization, examine a "tube" of dielectric parallel to **P**. The dipole moment of the tiny chunk shown in Fig. 4.12 is $P(Ad)$, where A is the cross-sectional area of the tube and d is the length of the chunk. In terms of the charge (q) at the end, this same dipole moment can be written qd. The bound charge that piles up at the right end of the tube is therefore

$$q = PA.$$

If the ends have been sliced off perpendicularly, the surface charge density is

$$\sigma_b = \frac{q}{A} = P.$$

For an oblique cut (Fig. 4.13), the *charge* is still the same, but $A = A_{\text{end}} \cos\theta$, so

$$\sigma_b = \frac{q}{A_{\text{end}}} = P\cos\theta = \mathbf{P} \cdot \hat{\mathbf{n}}.$$

The effect of the polarization, then, is to paint a bound charge $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ over the surface of the material. This is exactly what we found by more rigorous means in Sect. 4.2.1. But now we know where the bound charge *comes* from.

If the polarization is nonuniform, we get accumulations of bound charge *within* the material, as well as on the surface. A glance at Fig. 4.14 suggests that a diverging **P** results in a pileup of negative charge. Indeed, the net bound charge $\int \rho_b \, d\tau$

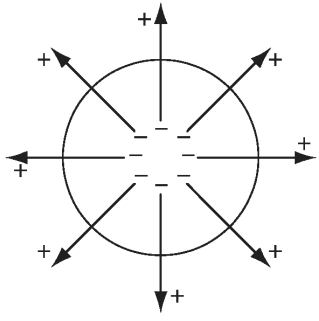


FIGURE 4.14

in a given volume is equal and opposite to the amount that has been pushed out through the surface. The latter (by the same reasoning we used before) is $\mathbf{P} \cdot \hat{\mathbf{n}}$ per unit area, so

$$\int_{\mathcal{V}} \rho_b \, d\tau = -\oint_{\mathcal{S}} \mathbf{P} \cdot d\mathbf{a} = -\int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{P}) \, d\tau.$$

Since this is true for *any* volume, we have

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P},$$

confirming, again, the more rigorous conclusion of Sect. 4.2.1.

Example 4.3. There is another way of analyzing the uniformly polarized sphere (Ex. 4.2), which nicely illustrates the idea of a bound charge. What we have, really, is *two* spheres of charge: a positive sphere and a negative sphere. Without polarization the two are superimposed and cancel completely. But when the material is uniformly polarized, all the plus charges move slightly *upward* (the z direction), and all the minus charges move slightly *downward* (Fig. 4.15). The two spheres no longer overlap perfectly: at the top there's a "cap" of leftover positive charge and at the bottom a cap of negative charge. This "leftover" charge is precisely the bound surface charge σ_b.

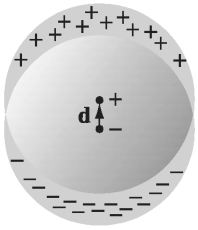


FIGURE 4.15

In Prob. 2.18, you calculated the field in the region of overlap between two uniformly charged spheres; the answer was

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{q\mathbf{d}}{R^3},$$

where q is the total charge of the positive sphere, $\mathbf{d}$ is the vector from the negative center to the positive center, and R is the radius of the sphere. We can express this in terms of the polarization of the sphere, $\mathbf{p} = q\mathbf{d} = (\frac{4}{3}\pi R^3)\mathbf{P}$, as

$$\mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}.$$

Meanwhile, for points *outside,* it is as though all the charge on each sphere were concentrated at the respective center. We have, then, a dipole, with potential

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}.$$

(Remember that $\mathbf{d}$ is some small fraction of an atomic radius; Fig. 4.15 is grossly exaggerated.) These answers agree, of course, with the results of Ex. 4.2.

Problem 4.13 A very long cylinder, of radius a, carries a uniform polarization $\mathbf{P}$ perpendicular to its axis. Find the electric field inside the cylinder. Show that the field *outside* the cylinder can be expressed in the form

$$\mathbf{E}(\mathbf{r}) = \frac{a^2}{2\epsilon_0 s^2}[2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}].$$

[*Careful:* I said "uniform," not "radial"!]

Problem 4.14 When you polarize a neutral dielectric, the charge moves a bit, but the *total* remains zero. This fact should be reflected in the bound charges σ_b and ρ_b. Prove from Eqs. 4.11 and 4.12 that the total bound charge vanishes.

4.2.3 ■ The Field Inside a Dielectric[4]

I have been sloppy about the distinction between "pure" dipoles and "physical" dipoles. In developing the theory of bound charges, I assumed we were working with the pure kind—indeed, I started with Eq. 4.8, the formula for the potential of a perfect dipole. And yet, an actual polarized dielectric consists of *physical* dipoles, albeit extremely tiny ones. What is more, I presumed to represent discrete molecular dipoles by a continuous density function $\mathbf{P}$. How can I justify this method? *Outside* the dielectric there is no real problem: here we are far away from the molecules (r is many times greater than the separation distance between plus and minus charges), so the dipole potential dominates overwhelmingly and the detailed "graininess" of the source is blurred by distance. *Inside* the dielectric, however, we can hardly pretend to be far from all the dipoles, and the procedure I used in Sect. 4.2.1 is open to serious challenge.

In fact, when you stop to think about it, the electric field inside matter must be fantastically complicated, on the microscopic level. If you happen to be very near an electron, the field is gigantic, whereas a short distance away it may be small or may point in a totally different direction. Moreover, an instant later, as the atoms move about, the field will have altered entirely. This true **microscopic** field would be utterly impossible to calculate, nor would it be of much interest if you could. Just as, for macroscopic purposes, we regard water as a continuous fluid, ignoring its molecular structure, so also we can ignore the microscopic

[4]This section can be skipped without loss of continuity.

bumps and wrinkles in the electric field inside matter, and concentrate on the **macroscopic** field. This is defined as the *average* field over regions large enough to contain many thousands of atoms (so that the uninteresting microscopic fluctuations are smoothed over), and yet small enough to ensure that we do not wash out any significant large-scale variations in the field. (In practice, this means we must average over regions much smaller than the dimensions of the object itself.) Ordinarily, the macroscopic field is what people *mean* when they speak of "the" field inside matter.[5]

It remains to show that the macroscopic field is what we actually obtain when we use the methods of Sect. 4.2.1. The argument is subtle, so hang on. Suppose I want to calculate the macroscopic field at some point $\mathbf{r}$ within a dielectric (Fig. 4.16). I know I must average the true (microscopic) field over an appropriate volume, so let me draw a small sphere about $\mathbf{r}$, of radius, say, a thousand times the size of a molecule. The macroscopic field at $\mathbf{r}$, then, consists of two parts: the average field over the sphere due to all charges *outside*, plus the average due to all charges *inside:*

$$\mathbf{E} = \mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}}.$$

You proved in Prob. 3.47(d) that the average field (over a sphere), produced by charges *outside*, is equal to the field they produce at the center, so $\mathbf{E}_{\text{out}}$ is the field at $\mathbf{r}$ due to the dipoles exterior to the sphere. These are far enough away that we can safely use Eq. 4.9:

$$V_{\text{out}} = \frac{1}{4\pi\epsilon_0} \int_{\text{outside}} \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\boldsymbol{\imath}}}{\imath^2} d\tau'. \tag{4.17}$$

The dipoles *inside* the sphere are too close to treat in this fashion. But fortunately all we need is their *average* field, and that, according to Eq. 3.105, is

$$\mathbf{E}_{\text{in}} = -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3},$$

regardless of the details of the charge distribution within the sphere. The only relevant quantity is the total dipole moment, $\mathbf{p} = (\frac{4}{3}\pi R^3)\,\mathbf{P}$:

$$\mathbf{E}_{\text{in}} = -\frac{1}{3\epsilon_0}\mathbf{P}. \tag{4.18}$$

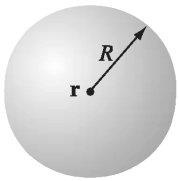


FIGURE 4.16

[5]In case the notion of macroscopic fields sounds suspicious to you, let me point out that you do *exactly* the same averaging whenever you speak of the *density* of a material.

Now, by assumption, the sphere is small enough that $\mathbf{P}$ does not vary significantly over its volume, so the term *left out* of the integral in Eq. 4.17 corresponds to the field at the center of a *uniformly* polarized sphere, to wit: $-(1/3\epsilon_0)\mathbf{P}$ (Eq. 4.14). But this is precisely what $\mathbf{E}_{\text{in}}$ (Eq. 4.18) puts back in! The macroscopic field, then, is given by the potential

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau', \tag{4.19}$$

where the integral runs over the *entire* volume of the dielectric. This is, of course, what we used in Sect. 4.2.1; without realizing it, we were correctly calculating the averaged, macroscopic field, for points inside the dielectric.

You may have to reread the last couple of paragraphs for the argument to sink in. Notice that it all revolves around the curious fact that the average field over *any* sphere (due to the charge inside) is the same as the field at the center of a *uniformly polarized* sphere with the same total dipole moment. This means that no matter how crazy the actual microscopic charge configuration, we can replace it by a nice smooth distribution of perfect dipoles, if all we want is the macroscopic (average) field. Incidentally, while the argument ostensibly relies on the spherical shape I chose to average over, the macroscopic field is certainly independent of the geometry of the averaging region, and this is reflected in the final answer, Eq. 4.19. Presumably one could reproduce the same argument for a cube or an ellipsoid or whatever—the calculation might be more difficult, but the conclusion would be the same.

4.3 ■ THE ELECTRIC DISPLACEMENT

4.3.1 ■ Gauss's Law in the Presence of Dielectrics

In Sect. 4.2 we found that the effect of polarization is to produce accumulations of (bound) charge, $\rho_b = -\nabla\cdot\mathbf{P}$ within the dielectric and $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ on the surface. The field due to polarization of the medium is just the field of this bound charge. We are now ready to put it all together: the field attributable to bound charge plus the field due to everything *else* (which, for want of a better term, we call **free charge**, ρ_f). The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever; any charge, in other words, that is *not* a result of polarization. Within the dielectric, the total charge density can be written:

$$\rho = \rho_b + \rho_f, \tag{4.20}$$

and Gauss's law reads

$$\epsilon_0\nabla\cdot\mathbf{E} = \rho = \rho_b + \rho_f = -\nabla\cdot\mathbf{P} + \rho_f,$$

where $\mathbf{E}$ is now the *total* field, not just that portion generated by polarization.

It is convenient to combine the two divergence terms:

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f.$$

The expression in parentheses, designated by the letter **D**,

$$\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P},} \tag{4.21}$$

is known as the **electric displacement**. In terms of **D**, Gauss's law reads

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f,} \tag{4.22}$$

or, in integral form,

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}, \tag{4.23}$$

where $Q_{f_{\text{enc}}}$ denotes the total free charge enclosed in the volume. This is a particularly useful way to express Gauss's law, in the context of dielectrics, because *it makes reference only to free charges,* and free charge is the stuff we control. Bound charge comes along for the ride: when we put the free charge in place, a certain polarization automatically ensues, by the mechanisms of Sect. 4.1, and this polarization produces the bound charge. In a typical problem, therefore, we know ρ_f, but we do not (initially) know ρ_b; Eq. 4.23 lets us go right to work with the information at hand. In particular, whenever the requisite symmetry is present, we can immediately calculate **D** by the standard Gauss's law methods.

Example 4.4. A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a (Fig. 4.17). Find the electric displacement.

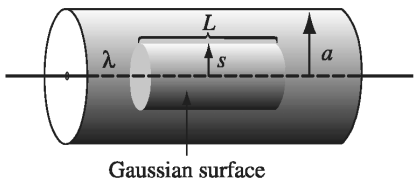


FIGURE 4.17

Solution

Drawing a cylindrical Gaussian surface, of radius s and length L, and applying Eq. 4.23, we find

$$D(2\pi s L) = \lambda L.$$

Therefore,

$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}. \tag{4.24}$$

Notice that this formula holds both within the insulation and outside it. In the latter region, $\mathbf{P} = 0$, so

$$\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}, \qquad \text{for } s > a.$$

Inside the rubber, the electric field cannot be determined, since we do not know **P**.

It may appear to you that I left out the surface bound charge σ_b in deriving Eq. 4.22, and in a sense that is true. We cannot apply Gauss's law precisely *at* the surface of a dielectric, for here ρ_b blows up,[6] taking the divergence of **E** with it. But everywhere *else* the logic is sound, and in fact if we picture the edge of the dielectric as having some finite thickness, within which the polarization tapers off to zero (probably a more realistic model than an abrupt cut-off anyway), then there *is* no surface bound charge; ρ_b varies rapidly but smoothly within this "skin," and Gauss's law can be safely applied *everywhere.* At any rate, the integral form (Eq. 4.23) is free from this "defect."

Problem 4.15 A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}},$$

where k is a constant and r is the distance from the center (Fig. 4.18). (There is no *free* charge in the problem.) Find the electric field in all three regions by two different methods:

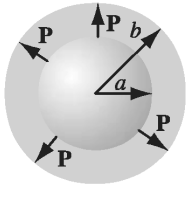


FIGURE 4.18

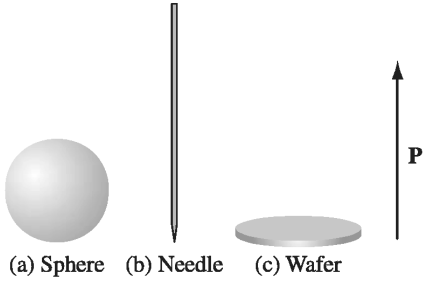


FIGURE 4.19

[6]The polarization drops abruptly to zero outside the material, so its *derivative* is a delta function (see Prob. 1.46). The surface bound charge *is* precisely this term—in this sense it is actually *included* in ρ_b, but we ordinarily prefer to handle it separately as σ_b.

(a) Locate all the bound charge, and use Gauss's law (Eq. 2.13) to calculate the field it produces.

(b) Use Eq. 4.23 to find **D**, and then get **E** from Eq. 4.21. [Notice that the second method is much faster, and it avoids any explicit reference to the bound charges.]

Problem 4.16 Suppose the field inside a large piece of dielectric is $\mathbf{E}_0$, so that the electric displacement is $\mathbf{D}_0 = \epsilon_0\mathbf{E}_0 + \mathbf{P}$.

(a) Now a small spherical cavity (Fig. 4.19a) is hollowed out of the material. Find the field at the center of the cavity in terms of $\mathbf{E}_0$ and **P**. Also find the displacement at the center of the cavity in terms of $\mathbf{D}_0$ and **P**. Assume the polarization is "frozen in," so it doesn't change when the cavity is excavated.

(b) Do the same for a long needle-shaped cavity running parallel to **P** (Fig. 4.19b).

(c) Do the same for a thin wafer-shaped cavity perpendicular to **P** (Fig. 4.19c).

Assume the cavities are small enough that **P**, $\mathbf{E}_0$, and $\mathbf{D}_0$ are essentially uniform. [*Hint:* Carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.]

4.3.2 ■ A Deceptive Parallel

Equation 4.22 looks just like Gauss's law, only the *total* charge density ρ is replaced by the *free* charge density ρ_f, and **D** is substituted for $\epsilon_0\mathbf{E}$. For this reason, you may be tempted to conclude that **D** is "just like" **E** (apart from the factor ϵ_0), except that its source is ρ_f instead of ρ: "To solve problems involving dielectrics, you just forget all about the bound charge—calculate the field as you ordinarily would, only call the answer **D** instead of **E**." This reasoning is seductive, but the conclusion is false; in particular, there is no "Coulomb's law" for **D**:

$$\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi}\int \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\rho_f(\mathbf{r}')\,d\tau'.$$

The parallel between **E** and **D** is more subtle than that.

For the divergence alone is insufficient to determine a vector field; you need to know the curl as well. One tends to forget this in the case of electrostatic fields because the curl of **E** is always zero. But the curl of **D** is *not* always zero.

$$\nabla \times \mathbf{D} = \epsilon_0(\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P}, \tag{4.25}$$

and there is no reason, in general, to suppose that the curl of **P** vanishes. Sometimes it does, as in Ex. 4.4 and Prob. 4.15, but more often it does not. The bar electret of Prob. 4.11 is a case in point: here there is no free charge anywhere, so if you really believe that the only source of **D** is ρ_f, you will be forced to conclude that $\mathbf{D} = \mathbf{0}$ everywhere, and hence that $\mathbf{E} = (-1/\epsilon_0)\mathbf{P}$ inside and $\mathbf{E} = \mathbf{0}$ outside the electret, which is obviously wrong. (I leave it for you to find the place where $\nabla \times \mathbf{P} \neq \mathbf{0}$ in this problem.) Because $\nabla \times \mathbf{D} \neq \mathbf{0}$, moreover, **D** cannot be expressed as the gradient of a scalar—there is no "potential" for **D**.

Advice: When you are asked to compute the electric displacement, first look for symmetry. If the problem exhibits spherical, cylindrical, or plane symmetry, then you can get **D** directly from Eq. 4.23 by the usual Gauss's law methods. (Evidently in such cases $\nabla \times \mathbf{P}$ is automatically zero, but since symmetry alone dictates the answer, you're not really obliged to worry about the curl.) If the requisite symmetry is absent, you'll have to think of another approach, and, in particular, you must *not* assume that **D** is determined exclusively by the free charge.

4.3.3 ■ Boundary Conditions

The electrostatic boundary conditions of Sect. 2.3.5 can be recast in terms of **D**. Equation 4.23 tells us the discontinuity in the component perpendicular to an interface:

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f, \tag{4.26}$$

while Eq. 4.25 gives the discontinuity in parallel components:

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}. \tag{4.27}$$

In the presence of dielectrics, these are sometimes more useful than the corresponding boundary conditions on **E** (Eqs. 2.31 and 2.32):

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma, \tag{4.28}$$

and

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = \mathbf{0}. \tag{4.29}$$

You might try applying them, for example, to Probs. 4.16 and 4.17.

Problem 4.17 For the bar electret of Prob. 4.11, make three careful sketches: one of **P**, one of **E**, and one of **D**. Assume L is about $2a$. [*Hint:* **E** lines terminate on charges; **D** lines terminate on *free* charges.]

4.4 ■ LINEAR DIELECTRICS

4.4.1 ■ Susceptibility, Permittivity, Dielectric Constant

In Sects. 4.2 and 4.3 we did not commit ourselves as to the *cause* of **P**; we dealt only with the *effects* of polarization. From the qualitative discussion of Sect. 4.1, though, we know that the polarization of a dielectric ordinarily results from an electric field, which lines up the atomic or molecular dipoles. For many substances, in fact, the polarization is *proportional* to the field, provided **E** is not too strong:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}. \tag{4.30}$$

The constant of proportionality, χ_e, is called the **electric susceptibility** of the medium (a factor of ϵ_0 has been extracted to make χ_e dimensionless). The value of χ_e depends on the microscopic structure of the substance in question (and also on external conditions such as temperature). I shall call materials that obey Eq. 4.30 **linear dielectrics.**[7]

Note that $\mathbf{E}$ in Eq. 4.30 is the *total* field; it may be due in part to free charges and in part to the polarization itself. If, for instance, we put a piece of dielectric into an external field $\mathbf{E}_0$, we cannot compute $\mathbf{P}$ directly from Eq. 4.30; the external field will polarize the material, and this polarization will produce its own field, which then contributes to the total field, and this in turn modifies the polarization, which . . . Breaking out of this infinite regress is not always easy. You'll see some examples in a moment. The simplest approach is to begin with the *displacement,* at least in those cases where $\mathbf{D}$ can be deduced directly from the free charge distribution.

In linear media we have

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E} = \epsilon_0(1 + \chi_e)\mathbf{E}, \tag{4.31}$$

so $\mathbf{D}$ is *also* proportional to $\mathbf{E}$:

$$\mathbf{D} = \epsilon\mathbf{E}, \tag{4.32}$$

where

$$\epsilon \equiv \epsilon_0(1 + \chi_e). \tag{4.33}$$

This new constant ϵ is called the **permittivity** of the material. (In vacuum, where there is no matter to polarize, the susceptibility is zero, and the permittivity is ϵ_0. That's why ϵ_0 is called the **permittivity of free space**. I dislike the term, for it suggests that the vacuum is just a special kind of linear dielectric, in which the permittivity happens to have the value $8.85 \times 10^{-12}\ \mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$.) If you remove a factor of ϵ_0, the remaining dimensionless quantity

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \tag{4.34}$$

is called the **relative permittivity**, or **dielectric constant**, of the material. Dielectric constants for some common substances are listed in Table 4.2. (Notice that ϵ_r is greater than 1, for all ordinary materials.) Of course, the permittivity and the dielectric constant do not convey any information that was not already available in the susceptibility, nor is there anything essentially new in Eq. 4.32; the *physics* of linear dielectrics is all contained in Eq. 4.30.[8]

[7] In modern optical applications, especially, *non*linear materials have become increasingly important. For these there is a second term in the formula for $\mathbf{P}$ as a function of $\mathbf{E}$—typically a *cubic* term. In general, Eq. 4.30 can be regarded as the first (nonzero) term in the Taylor expansion of $\mathbf{P}$ in powers of $\mathbf{E}$.

[8] As long as we are engaged in this orgy of unnecessary terminology and notation, I might as well mention that formulas for $\mathbf{D}$ in terms of $\mathbf{E}$ (Eq. 4.32, in the case of linear dielectrics) are called **constitutive relations**.

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7-5.9
Neon	1.00013	Salt	5.9
Hydrogen (H_2)	1.000254	Silicon	11.7
Argon	1.000517	Methanol	33.0
Air (dry)	1.000536	Water	80.1
Nitrogen (N_2)	1.000548	Ice (-30° C)	104
Water vapor (100° C)	1.00589	$KTaNbO_3$ (0° C)	34,000

TABLE 4.2 Dielectric Constants (unless otherwise specified, values given are for 1 atm, 20° C). *Data from Handbook of Chemistry and Physics,* 91st ed. (Boca Raton: CRC Press, 2010).

Example 4.5. A metal sphere of radius a carries a charge Q (Fig. 4.20). It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).

Solution

To compute V, we need to know $\mathbf{E}$; to find $\mathbf{E}$, we might first try to locate the bound charge; we could get the bound charge from $\mathbf{P}$, but we can't calculate $\mathbf{P}$ unless we already know $\mathbf{E}$ (Eq. 4.30). We seem to be in a bind. What we *do* know is the *free* charge Q, and fortunately the arrangement is spherically symmetric, so let's begin by calculating $\mathbf{D}$, using Eq. 4.23:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for all points } r > a.$$

(Inside the metal sphere, of course, $\mathbf{E} = \mathbf{P} = \mathbf{D} = \mathbf{0}$.) Once we know $\mathbf{D}$, it is a trivial matter to obtain $\mathbf{E}$, using Eq. 4.32:

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

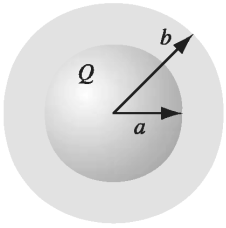


FIGURE 4.20

The potential at the center is therefore

$$V = -\int_{\infty}^{0} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{b}\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)dr - \int_{b}^{a}\left(\frac{Q}{4\pi\epsilon r^2}\right)dr - \int_{a}^{0}(0)\,dr$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right).$$

As it turns out, it was not necessary for us to compute the polarization or the bound charge explicitly, though this can easily be done:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}},$$

in the dielectric, and hence

$$\rho_b = -\nabla\cdot\mathbf{P} = 0,$$

while

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface,} \\ \dfrac{-\epsilon_0\chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface.} \end{cases}$$

Notice that the surface bound charge at a is *negative* ($\hat{\mathbf{n}}$ points outward *with respect to the dielectric,* which is $+\hat{\mathbf{r}}$ at b but $-\hat{\mathbf{r}}$ at a). This is natural, since the charge on the metal sphere attracts its opposite in all the dielectric molecules. It is this layer of negative charge that reduces the field, within the dielectric, from $1/4\pi\epsilon_0(Q/r^2)\hat{\mathbf{r}}$ to $1/4\pi\epsilon(Q/r^2)\hat{\mathbf{r}}$. In this respect, a dielectric is rather like an imperfect conductor: on a *conducting* shell the induced surface charge would be such as to cancel the field of Q *completely* in the region $a < r < b$; the dielectric does the best it can, but the cancellation is only partial.

You might suppose that linear dielectrics escape the defect in the parallel between **E** and **D**. Since **P** and **D** are now proportional to **E**, does it not follow that their curls, like **E**'s, must vanish? Unfortunately, it does *not,* for the line integral of **P** around a closed path that *straddles the boundary between one type of material and another* need not be zero, even though the integral of **E** around the same loop *must* be. The reason is that the proportionality factor $\epsilon_0\chi_e$ is different on the two sides. For instance, at the interface between a polarized dielectric and the vacuum (Fig. 4.21), **P** is zero on one side but not on the other. Around this

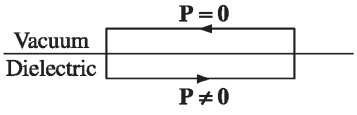


FIGURE 4.21

loop $\oint \mathbf{P} \cdot d\mathbf{l} \neq 0$, and hence, by Stokes' theorem, the curl of $\mathbf{P}$ cannot vanish everywhere within the loop (in fact, it is *infinite* at the boundary).[9]

Of course, if the space is *entirely* filled with a homogeneous[10] linear dielectric, then this objection is void; in this rather special circumstance

$$\nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \times \mathbf{D} = \mathbf{0},$$

so $\mathbf{D}$ can be found from the free charge just as though the dielectric were not there:

$$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{vac}},$$

where $\mathbf{E}_{\text{vac}}$ is the field the same free charge distribution would produce in the absence of any dielectric. According to Eqs. 4.32 and 4.34, therefore,

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{1}{\epsilon_r}\mathbf{E}_{\text{vac}}. \tag{4.35}$$

Conclusion: When all space is filled with a homogeneous linear dielectric, the field everywhere is simply reduced by a factor of one over the dielectric constant. (Actually, it is not necessary for the dielectric to fill *all* space: in regions where the field is zero anyway, it can hardly matter whether the dielectric is present or not, since there's no polarization in any event.)

For example, if a free charge q is embedded in a large dielectric, the field it produces is

$$\mathbf{E} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{\mathbf{r}} \tag{4.36}$$

(that's ϵ, not ϵ_0), and the force it exerts on nearby charges is reduced accordingly. But it's not that there is anything wrong with Coulomb's law; rather, the polarization of the medium partially "shields" the charge, by surrounding it with bound charge of the opposite sign (Fig. 4.22).[11]

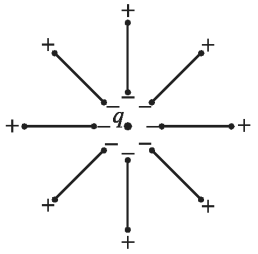


FIGURE 4.22

[9] Putting that argument in differential form, Eq. 4.30 and product rule 7 yield $\nabla \times \mathbf{P} = -\epsilon_0 \mathbf{E} \times (\nabla \chi_e)$, so the problem arises when $\nabla \chi_e$ is not parallel to $\mathbf{E}$.

[10] A **homogeneous** medium is one whose properties (in this case the susceptibility) do not vary with position.

[11] In *quantum* electrodynamics, the vacuum itself can be polarized, and this means that the effective (or "renormalized") charge of the electron, as you might measure it in the laboratory, is not its true ("bare") value, and in fact depends slightly on how far away you are!

Example 4.6. A parallel-plate capacitor (Fig. 4.23) is filled with insulating material of dielectric constant ϵ_r. What effect does this have on its capacitance?

Solution

Since the field is confined to the space between the plates, the dielectric will reduce **E**, and hence also the potential difference V, by a factor $1/\epsilon_r$. Accordingly, the capacitance $C = Q/V$ is *increased by a factor of the dielectric constant,*

$$C = \epsilon_r C_{\text{vac}}. \tag{4.37}$$

This is, in fact, a common way to beef up a capacitor.

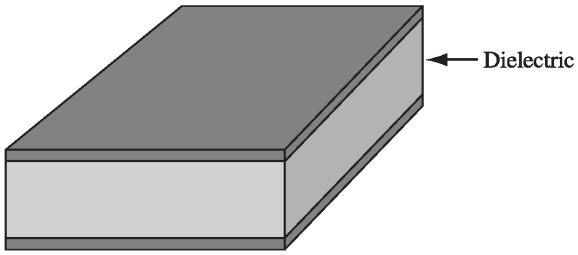


FIGURE 4.23

A *crystal* is generally easier to polarize in some directions than in others,[12] and in this case Eq. 4.30 is replaced by the general linear relation

$$\left.\begin{aligned} P_x &= \epsilon_0(\chi_{e_{xx}} E_x + \chi_{e_{xy}} E_y + \chi_{e_{xz}} E_z) \\ P_y &= \epsilon_0(\chi_{e_{yx}} E_x + \chi_{e_{yy}} E_y + \chi_{e_{yz}} E_z) \\ P_z &= \epsilon_0(\chi_{e_{zx}} E_x + \chi_{e_{zy}} E_y + \chi_{e_{zz}} E_z) \end{aligned}\right\}, \tag{4.38}$$

just as Eq. 4.1 was superseded by Eq. 4.3 for asymmetrical molecules. The nine coefficients, $\chi_{e_{xx}}, \chi_{e_{xy}}, \ldots$, constitute the **susceptibility tensor.**

Problem 4.18 The space between the plates of a parallel-plate capacitor (Fig. 4.24) is filled with two slabs of linear dielectric material. Each slab has thickness a, so the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate $-\sigma$.

[12] A medium is said to be **isotropic** if its properties (such as susceptibility) are the same in all directions. Thus Eq. 4.30 is the special case of Eq. 4.38 that holds for isotropic media. Physicists tend to be sloppy with their language, and unless otherwise indicated the term "linear dielectric" implies "isotropic linear dielectric," and suggests "homogeneous isotropic linear dielectric." But technically, "linear" just means that at *any given point*, and for **E** in a *given direction*, the components of **P** are proportional to E—the proportionality factor could vary with position and/or direction.

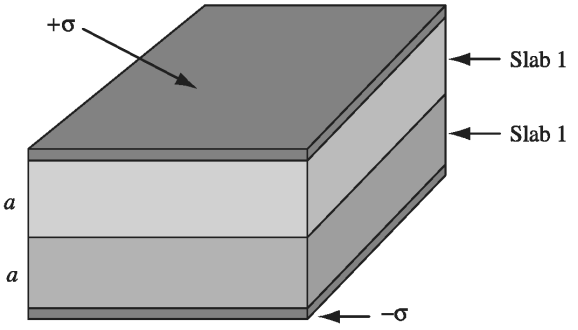


FIGURE 4.24

(a) Find the electric displacement **D** in each slab.

(b) Find the electric field **E** in each slab.

(c) Find the polarization **P** in each slab.

(d) Find the potential difference between the plates.

(e) Find the location and amount of all bound charge.

(f) Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (b).

Problem 4.19 Suppose you have enough linear dielectric material, of dielectric constant ϵ_r, to *half*-fill a parallel-plate capacitor (Fig. 4.25). By what fraction is the capacitance increased when you distribute the material as in Fig. 4.25(a)? How about Fig. 4.25(b)? For a given potential difference V between the plates, find **E**, **D**, and **P**, in each region, and the free and bound charge on all surfaces, for both cases.

Problem 4.20 A sphere of linear dielectric material has embedded in it a uniform free charge density ρ. Find the potential at the center of the sphere (relative to infinity), if its radius is R and the dielectric constant is ϵ_r.

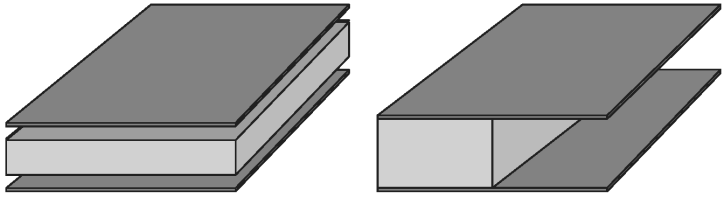


FIGURE 4.25

Problem 4.21 A certain coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius c (Fig. 4.26). The space between is partially filled (from b out to c) with material of dielectric constant ϵ_r, as shown. Find the capacitance per unit length of this cable.

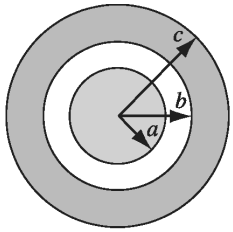


FIGURE 4.26

4.4.2 ■ Boundary Value Problems with Linear Dielectrics

In a (homogeneous isotropic) linear dielectric, the bound charge density (ρ_b) is proportional to the free charge density (ρ_f):[13]

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\epsilon_0 \frac{\chi_e}{\epsilon} \mathbf{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right) \rho_f. \tag{4.39}$$

In particular, unless free charge is actually embedded in the material, $\rho = 0$, and any net charge must reside at the surface. Within such a dielectric, then, the potential obeys Laplace's equation, and all the machinery of Chapter 3 carries over. It is convenient, however, to rewrite the boundary conditions in a way that makes reference only to the free charge. Equation 4.26 says

$$\epsilon_{\text{above}} E^{\perp}_{\text{above}} - \epsilon_{\text{below}} E^{\perp}_{\text{below}} = \sigma_f, \tag{4.40}$$

or (in terms of the potential),

$$\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f, \tag{4.41}$$

whereas the potential itself is, of course, continuous (Eq. 2.34):

$$V_{\text{above}} = V_{\text{below}}. \tag{4.42}$$

[13] This does not apply to the surface charge (σ_b), because χ_e is not independent of position (obviously) at the boundary.

Example 4.7. A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$ (Fig. 4.27). Find the electric field inside the sphere.

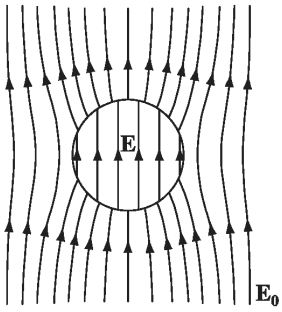


FIGURE 4.27

Solution

This is reminiscent of Ex. 3.8, in which an uncharged *conducting* sphere was introduced into a uniform field. In that case, the field of the induced charge canceled $\mathbf{E}_0$ within the sphere; in a *dielectric,* the cancellation (from the bound charge) is incomplete.

Our problem is to solve Laplace's equation, for $V_{\text{in}}(r, \theta)$ when $r \leq R$, and $V_{\text{out}}(r, \theta)$ when $r \geq R$, subject to the boundary conditions

$$\left.\begin{aligned} &\text{(i)} && V_{\text{in}} = V_{\text{out}}, && \text{at } r = R, \\ &\text{(ii)} && \epsilon \frac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \frac{\partial V_{\text{out}}}{\partial r}, && \text{at } r = R, \\ &\text{(iii)} && V_{\text{out}} \to -E_0 r \cos\theta, && \text{for } r \gg R. \end{aligned}\right\} \tag{4.43}$$

(The second of these follows from Eq. 4.41, since there is no free charge at the surface.) Inside the sphere, Eq. 3.65 says

$$V_{\text{in}}(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta); \tag{4.44}$$

outside the sphere, in view of (iii), we have

$$V_{\text{out}}(r, \theta) = -E_0 r \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{4.45}$$

Boundary condition (i) requires that

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -E_0 R \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta),$$

so[14]

$$\left.\begin{aligned} A_l R^l &= \frac{B_l}{R^{l+1}}, \quad \text{for } l \neq 1, \\ A_1 R &= -E_0 R + \frac{B_1}{R^2}. \end{aligned}\right\} \tag{4.46}$$

Meanwhile, condition (ii) yields

$$\epsilon_r \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -E_0 \cos\theta - \sum_{l=0}^{\infty} \frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta),$$

so

$$\left.\begin{aligned} \epsilon_r l A_l R^{l-1} &= -\frac{(l+1)B_l}{R^{l+2}}, \text{ for } l \neq 1, \\ \epsilon_r A_1 &= -E_0 - \frac{2B_1}{R^3}. \end{aligned}\right\} \tag{4.47}$$

It follows that

$$\left.\begin{aligned} A_l = B_l = 0, &\qquad \text{for } l \neq 1, \\ A_1 = -\frac{3}{\epsilon_r + 2} E_0 &\quad B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 2} R^3 E_0. \end{aligned}\right\} \tag{4.48}$$

Evidently

$$V_{\text{in}}(r, \theta) = -\frac{3E_0}{\epsilon_r + 2} r\cos\theta = -\frac{3E_0}{\epsilon_r + 2} z,$$

and hence the field inside the sphere is (surprisingly) *uniform*:

$$\mathbf{E} = \frac{3}{\epsilon_r + 2} \mathbf{E}_0. \tag{4.49}$$

Example 4.8. Suppose the entire region below the plane $z = 0$ in Fig. 4.28 is filled with uniform linear dielectric material of susceptibility χ_e. Calculate the force on a point charge q situated a distance d above the origin.

[14]Remember, $P_1(\cos\theta) = \cos\theta$, and the coefficients must be equal for each l, as you could prove by multiplying by $P_{l'}(\cos\theta)\sin\theta$, integrating from 0 to π, and invoking the orthogonality of the Legendre polynomials (Eq. 3.68).

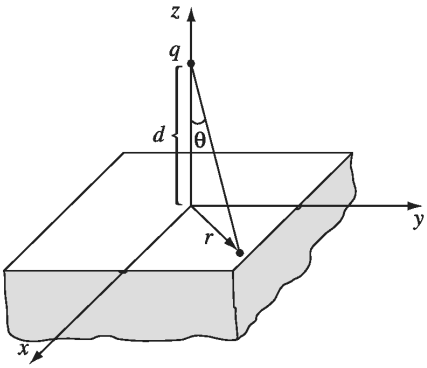


FIGURE 4.28

Solution

The surface bound charge on the xy plane is of opposite sign to q, so the force will be attractive. (In view of Eq. 4.39, there is no volume bound charge.) Let us first calculate σ_b, using Eqs. 4.11 and 4.30.[15]

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P_z = \epsilon_0 \chi_e E_z,$$

where E_z is the z-component of the total field just inside the dielectric, at $z = 0$. This field is due in part to q and in part to the bound charge itself. From Coulomb's law, the former contribution is

$$-\frac{1}{4\pi\epsilon_0}\frac{q}{(r^2+d^2)}\cos\theta = -\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}},$$

where $r = \sqrt{x^2+y^2}$ is the distance from the origin. The z component of the field of the bound charge, meanwhile, is $-\sigma_b/2\epsilon_0$ (see footnote after Eq. 2.33). Thus

$$\sigma_b = \epsilon_0 \chi_e \left[-\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} - \frac{\sigma_b}{2\epsilon_0} \right],$$

which we can solve for σ_b:

$$\sigma_b = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(r^2+d^2)^{3/2}}. \tag{4.50}$$

Apart from the factor $\chi_e/(\chi_e+2)$, this is exactly the same as the induced charge on an infinite *conducting* plane under similar circumstances (Eq. 3.10).[16] Evidently the *total* bound charge is

$$q_b = -\left(\frac{\chi_e}{\chi_e+2}\right)q. \tag{4.51}$$

[15]This method mimics Prob. 3.38.

[16]For some purposes a conductor can be regarded as the limiting case of a linear dielectric, with $\chi_e \to \infty$. This is often a useful check—try applying it to Exs. 4.5, 4.6, and 4.7.

We could, of course, obtain the field of σ_b by direct integration

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right)\sigma_b\, da.$$

But, as in the case of the conducting plane, there is a nicer solution by the method of images. Indeed, if we replace the dielectric by a single point charge q_b at the image position $(0, 0, -d)$, we have

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_b}{\sqrt{x^2+y^2+(z+d)^2}}\right], \tag{4.52}$$

in the region $z > 0$. Meanwhile, a charge $(q + q_b)$ at $(0, 0, d)$ yields the potential

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q+q_b}{\sqrt{x^2+y^2+(z-d)^2}}\right], \tag{4.53}$$

for the region $z < 0$. Taken together, Eqs. 4.52 and 4.53 constitute a function that satisfies Poisson's equation with a point charge q at $(0, 0, d)$, which goes to zero at infinity, which is continuous at the boundary $z = 0$, and whose normal derivative exhibits the discontinuity appropriate to a surface charge σ_b at $z = 0$:

$$-\epsilon_0\left(\left.\frac{\partial V}{\partial z}\right|_{z=0^+} - \left.\frac{\partial V}{\partial z}\right|_{z=0^-}\right) = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(x^2+y^2+d^2)^{3/2}}.$$

Accordingly, this is the correct potential for our problem. In particular, the force on q is:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_b}{(2d)^2}\hat{\mathbf{z}} = -\frac{1}{4\pi\epsilon_0}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{q^2}{4d^2}\hat{\mathbf{z}}. \tag{4.54}$$

I do not claim to have provided a compelling *motivation* for Eqs. 4.52 and 4.53—like all image solutions, this one owes its justification to the fact that it *works:* it solves Poisson's equation, and it meets the boundary conditions. Still, discovering an image solution is not entirely a matter of guesswork. There are at least two "rules of the game": (1) You must never put an image charge into the region where you're computing the potential. (Thus Eq. 4.52 gives the potential for $z > 0$, but this image charge q_b is at $z = -d$; when we turn to the region $z < 0$ (Eq. 4.53), the image charge $(q + q_b)$ is at $z = +d$.) (2) The image charges must add up to the correct total in each region. (That's how I knew to use q_b to account for the charge in the region $z \leq 0$, and $(q + q_b)$ to cover the region $z \geq 0$.)

Problem 4.22 A very long cylinder of linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the resulting field within the cylinder. (The radius is a, the susceptibility χ_e, and the axis is perpendicular to $\mathbf{E}_0$.)

Problem 4.23 Find the field inside a sphere of linear dielectric material in an otherwise uniform electric field $\mathbf{E}_0$ (Ex. 4.7) by the following method of successive approximations: First pretend the field inside is just $\mathbf{E}_0$, and use Eq. 4.30 to write down the resulting polarization $\mathbf{P}_0$. This polarization generates a field of its own, $\mathbf{E}_1$ (Ex. 4.2), which in turn modifies the polarization by an amount $\mathbf{P}_1$, which further changes the field by an amount $\mathbf{E}_2$, and so on. The resulting field is $\mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \cdots$. Sum the series, and compare your answer with Eq. 4.49.

Problem 4.24 An uncharged conducting sphere of radius a is coated with a thick insulating shell (dielectric constant ϵ_r) out to radius b. This object is now placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the electric field in the insulator.

! **Problem 4.25** Suppose the region *above* the xy plane in Ex. 4.8 is *also* filled with linear dielectric but of a different susceptibility χ_e'. Find the potential everywhere.

4.4.3 ■ Energy in Dielectric Systems

It takes work to charge up a capacitor (Eq. 2.55):

$$W = \tfrac{1}{2}CV^2.$$

If the capacitor is filled with linear dielectric, its capacitance exceeds the vacuum value by a factor of the dielectric constant,

$$C = \epsilon_r C_{\text{vac}},$$

as we found in Ex. 4.6. Evidently the work necessary to charge a dielectric-filled capacitor is increased by the same factor. The reason is pretty clear: you have to pump on more (free) charge, to achieve a given potential, because part of the field is canceled off by the bound charges.

In Chapter 2, I derived a general formula for the energy stored in any electrostatic system (Eq. 2.45):

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau. \tag{4.55}$$

The case of the dielectric-filled capacitor suggests that this should be changed to

$$W = \frac{\epsilon_0}{2}\int \epsilon_r E^2\, d\tau = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau,$$

in the presence of linear dielectrics. To *prove* it, suppose the dielectric material is fixed in position, and we bring in the free charge, a bit at a time. As ρ_f is increased by an amount $\Delta\rho_f$, the polarization will change and with it the bound charge distribution; but we're interested only in the work done on the incremental *free* charge:

$$\Delta W = \int (\Delta\rho_f) V\, d\tau. \tag{4.56}$$

Since $\nabla \cdot \mathbf{D} = \rho_f$, $\Delta\rho_f = \nabla \cdot (\Delta\mathbf{D})$, where $\Delta\mathbf{D}$ is the resulting change in $\mathbf{D}$, so

$$\Delta W = \int [\nabla \cdot (\Delta\mathbf{D})] V \, d\tau.$$

Now

$$\nabla \cdot [(\Delta\mathbf{D})V] = [\nabla \cdot (\Delta\mathbf{D})]V + \Delta\mathbf{D} \cdot (\nabla V),$$

and hence (integrating by parts):

$$\Delta W = \int \nabla \cdot [(\Delta\mathbf{D})V] \, d\tau + \int (\Delta\mathbf{D}) \cdot \mathbf{E} \, d\tau.$$

The divergence theorem turns the first term into a surface integral, which vanishes if we integrate over all space. Therefore, the work done is equal to

$$\Delta W = \int (\Delta\mathbf{D}) \cdot \mathbf{E} \, d\tau. \tag{4.57}$$

So far, this applies to *any* material. Now, if the medium is a linear dielectric, then $\mathbf{D} = \epsilon\mathbf{E}$, so

$$\tfrac{1}{2}\Delta(\mathbf{D} \cdot \mathbf{E}) = \tfrac{1}{2}\Delta(\epsilon E^2) = \epsilon(\Delta\mathbf{E}) \cdot \mathbf{E} = (\Delta\mathbf{D}) \cdot \mathbf{E}$$

(for infinitesimal increments). Thus

$$\Delta W = \Delta\left(\frac{1}{2}\int \mathbf{D} \cdot \mathbf{E} \, d\tau\right).$$

The total work done, then, as we build the free charge up from zero to the final configuration, is

$$W = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E} \, d\tau, \tag{4.58}$$

as anticipated.[17]

It may puzzle you that Eq. 4.55, which we derived quite generally in Chapter 2, does not seem to apply in the presence of dielectrics, where it is replaced by Eq. 4.58. The point is not that one or the other of these equations is *wrong,* but rather that they address somewhat different questions. The distinction is subtle, so let's go right back to the beginning: What do we *mean* by "the energy of a system"? *Answer:* It is the work required to assemble the system. Very

[17]In case you are wondering why I did not do this more simply by the method of Sect. 2.4.3, starting with $W = \frac{1}{2}\int \rho_f V \, d\tau$, the reason is that *this* formula is untrue, in general. Study the derivation of Eq. 2.42, and you will see that it applies only to the *total* charge. For *linear* dielectrics it happens to hold for the free charge alone, but this is scarcely obvious a priori and, in fact, is most easily confirmed by working backward from Eq. 4.58.

well—but when dielectrics are involved, there are two quite different ways one might construe this process:

1. We bring in all the charges (free *and* bound), one by one, with tweezers, and glue each one down in its proper final location. If *this* is what you mean by "assemble the system," then Eq. 4.55 is your formula for the energy stored. Notice, however, that this will *not* include the work involved in stretching and twisting the dielectric molecules (if we picture the positive and negative charges as held together by tiny springs, it does not include the spring energy, $\frac{1}{2}kx^2$, associated with polarizing each molecule).[18]

2. With the unpolarized dielectric in place, we bring in the *free* charges, one by one, allowing the dielectric to respond as it sees fit. If *this* is what you mean by "assemble the system" (and ordinarily it *is,* since free charge is what we actually push around), then Eq. 4.58 is the formula you want. In this case the "spring" energy *is* included, albeit indirectly, because the force you must apply to the *free* charge depends on the disposition of the *bound* charge; as you move the free charge, you are automatically stretching those "springs."

Example 4.9. A sphere of radius R is filled with material of dielectric constant ϵ_r and uniform embedded free charge ρ_f. What is the energy of this configuration?

Solution

From Gauss's law (in the form of Eq. 4.23), the displacement is

$$\mathbf{D}(r) = \begin{cases} \dfrac{\rho_f}{3}\mathbf{r} & (r < R), \\ \dfrac{\rho_f}{3}\dfrac{R^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases}$$

So the electric field is

$$\mathbf{E}(r) = \begin{cases} \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & (r < R), \\ \dfrac{\rho_f}{3\epsilon_0}\dfrac{R^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases}$$

The purely *electrostatic* energy (Eq. 4.55) is

$$W_1 = \frac{\epsilon_0}{2}\left[\left(\frac{\rho_f}{3\epsilon_0\epsilon_r}\right)^2 \int_0^R r^2\, 4\pi r^2\, dr + \left(\frac{\rho_f}{3\epsilon_0}\right)^2 R^6 \int_R^\infty \frac{1}{r^4}\, 4\pi r^2\, dr\right]$$
$$= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r^2} + 1\right).$$

[18]The "spring" itself may be electrical in nature, but it is still not included in Eq. 4.55, if $\mathbf{E}$ is taken to be the *macroscopic* field.

But the *total* energy (Eq. 4.58) is

$$W_2 = \frac{1}{2}\left[\left(\frac{\rho_f}{3}\right)\left(\frac{\rho_f}{3\epsilon_0\epsilon_r}\right)\int_0^R r^2\, 4\pi r^2\, dr + \left(\frac{\rho_f R^3}{3}\right)\left(\frac{\rho_f R^3}{3\epsilon_0}\right)\int_R^\infty \frac{1}{r^4}\, 4\pi r^2\, dr\right]$$
$$= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r} + 1\right).$$

Notice that $W_1 < W_2$—that's because W_1 does not include the energy involved in stretching the molecules.

Let's check that W_2 is the work done on the *free* charge in assembling the system. We start with the (uncharged, unpolarized) dielectric sphere, and bring in the free charge in infinitesimal installments (dq), filling out the sphere layer by layer. When we have reached radius r', the electric field is

$$\mathbf{E}(r) = \begin{cases} \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & (r < r'), \\ \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\dfrac{r'^3}{r^2}\hat{\mathbf{r}} & (r' < r < R), \\ \dfrac{\rho_f}{3\epsilon_0}\dfrac{r'^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases}$$

The work required to bring the next dq in from infinity to r' is

$$dW = -dq\left[\int_\infty^R \mathbf{E}\cdot d\mathbf{l} + \int_R^{r'} \mathbf{E}\cdot d\mathbf{l}\right]$$
$$= -dq\left[\frac{\rho_f r'^3}{3\epsilon_0}\int_\infty^R \frac{1}{r^2}\, dr + \frac{\rho_f r'^3}{3\epsilon_0\epsilon_r}\int_R^{r'} \frac{1}{r^2}\, dr\right]$$
$$= \frac{\rho_f r'^3}{3\epsilon_0}\left[\frac{1}{R} + \frac{1}{\epsilon_r}\left(\frac{1}{r'} - \frac{1}{R}\right)\right] dq.$$

This increases the radius (r'):

$$dq = \rho_f 4\pi r'^2\, dr',$$

so the *total* work done, in going from $r' = 0$ to $r' = R$, is

$$W = \frac{4\pi\rho_f^2}{3\epsilon_0}\left[\frac{1}{R}\left(1 - \frac{1}{\epsilon_r}\right)\int_0^R r'^5\, dr' + \frac{1}{\epsilon_r}\int_0^R r'^4\, dr'\right]$$
$$= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r} + 1\right) = W_2. \checkmark$$

Evidently the energy "stored in the springs" is

$$W_{\text{spring}} = W_2 - W_1 = \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2 R^5\,(\epsilon_r - 1)\,.$$

I would like to confirm this in an explicit model. Picture the dielectric as a collection of tiny proto-dipoles, each consisting of $+q$ and $-q$ attached to a spring of constant k and equilibrium length 0, so in the absence of any field the positive and negative ends coincide. One end of each dipole is nailed in position (like the nuclei in a solid), but the other end is free to move in response to any imposed field. Let $d\tau$ be the volume assigned to each proto-dipole (the dipole itself may occupy only a small portion of this space).

With the field turned on, the electric force on the free end is balanced by the spring force;[19] the charges separate by a distance d: $qE = kd$. In our case

$$\mathbf{E} = \frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r}.$$

The resulting dipole moment is $p = qd$, and the polarization is $P = p/d\tau$, so

$$k = \frac{\rho_f}{3\epsilon_0\epsilon_r d^2} P r\, d\tau.$$

The energy of this particular spring is

$$dW_{\text{spring}} = \frac{1}{2}kd^2 = \frac{\rho_f}{6\epsilon_0\epsilon_r} P r\, d\tau,$$

and hence the total is

$$W_{\text{spring}} = \frac{\rho_f}{6\epsilon_0\epsilon_r}\int P r\, d\tau.$$

Now

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \epsilon_0\chi_e\frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} = \frac{(\epsilon_r - 1)\rho_f}{3\epsilon_r}\mathbf{r},$$

so

$$W_{\text{spring}} = \frac{\rho_f}{6\epsilon_0\epsilon_r}\frac{(\epsilon_r - 1)\rho_f}{3\epsilon_r}4\pi\int_0^R r^4\, dr = \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2 R^5\,(\epsilon_r - 1)\,,$$

and it works out perfectly.

It is sometimes alleged that Eq. 4.58 represents the energy even for *non*linear dielectrics, but this is false: To proceed beyond Eq. 4.57, one must assume linearity. In fact, for *dissipative* systems the whole notion of "stored energy" loses its meaning, because the work done depends not only on the final configuration but on *how it got there.* If the molecular "springs" are allowed to have some

[19]Note that the "spring" here is a surrogate for whatever holds the molecule together—it *includes* the electrical attraction of the other end. If it bothers you that the force is taken to be proportional to the separation, look again at Example 4.1.

friction, for instance, then W_{spring} can be made as large as you like, by assembling the charges in such a way that the spring is obliged to expand and contract many times before reaching its final state. In particular, you get nonsensical results if you try to apply Eq. 4.58 to electrets, with frozen-in polarization (see Prob. 4.27).

Problem 4.26 A spherical conductor, of radius a, carries a charge Q (Fig. 4.29). It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration (Eq. 4.58).

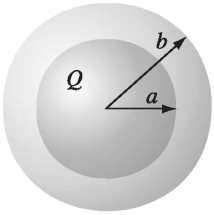


FIGURE 4.29

Problem 4.27 Calculate W, using both Eq. 4.55 and Eq. 4.58, for a sphere of radius R with frozen-in uniform polarization $\mathbf{P}$ (Ex. 4.2). Comment on the discrepancy. Which (if either) is the "true" energy of the system?

4.4.4 ■ Forces on Dielectrics

Just as a conductor is attracted into an electric field (Eq. 2.51), so too is a dielectric—and for essentially the same reason: the bound charge tends to accumulate near the free charge of the opposite sign. But the calculation of forces on dielectrics can be surprisingly tricky. Consider, for example, the case of a slab of linear dielectric material, partially inserted between the plates of a parallel-plate capacitor (Fig. 4.30). We have always pretended that the field is uniform inside a parallel-plate capacitor, and zero outside. If this were literally true, there would be no net force on the dielectric at all, since the field everywhere would be perpendicular to the plates. However, there is in reality a **fringing field** around the edges, which for most purposes can be ignored but in this case is responsible for the whole effect. (Indeed, the field *could* not terminate abruptly at the edge of the capacitor, for if it did, the line integral of $\mathbf{E}$ around the closed loop shown in Fig. 4.31 would not be zero.) It is this nonuniform fringing field that pulls the dielectric into the capacitor.

Fringing fields are notoriously difficult to calculate; luckily, we can avoid this altogether, by the following ingenious method.[20] Let W be the energy of the

[20]For a direct calculation from the fringing fields, see E. R. Dietz, *Am. J. Phys.* **72**, 1499 (2004).

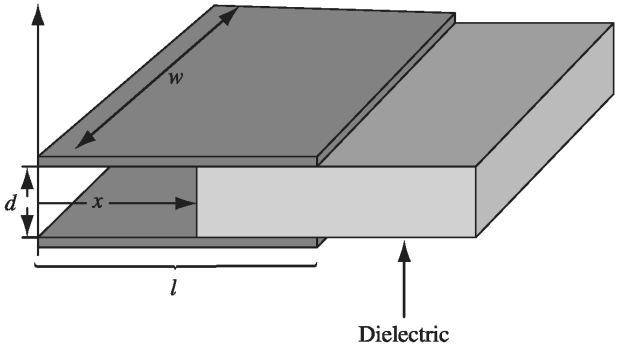


FIGURE 4.30

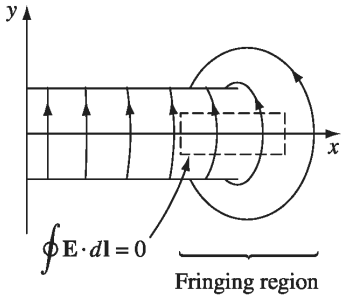


FIGURE 4.31

system—it depends, of course, on the amount of overlap. If I pull the dielectric out an infinitesimal distance dx, the energy is changed by an amount equal to the work done:

$$dW = F_{\text{me}}\, dx, \tag{4.59}$$

where F_{me} is the force I must exert, to counteract the electrical force F on the dielectric: $F_{\text{me}} = -F$. Thus the electrical force on the slab is

$$F = -\frac{dW}{dx}. \tag{4.60}$$

Now, the energy stored in the capacitor is

$$W = \tfrac{1}{2}CV^2, \tag{4.61}$$

and the capacitance in this case is

$$C = \frac{\epsilon_0 w}{d}(\epsilon_r l - \chi_e x), \tag{4.62}$$

where l is the length of the plates (Fig. 4.30). Let's assume that the total charge on the plates ($Q = CV$) is held constant, as the dielectric moves. In terms of Q,

$$W = \frac{1}{2}\frac{Q^2}{C}, \tag{4.63}$$

so

$$F = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}. \tag{4.64}$$

But

$$\frac{dC}{dx} = -\frac{\epsilon_0 \chi_e w}{d},$$

and hence

$$F = -\frac{\epsilon_0 \chi_e w}{2d}V^2. \tag{4.65}$$

(The minus sign indicates that the force is in the negative x direction; the dielectric is pulled *into* the capacitor.)

It is a common error to use Eq. 4.61 (with V constant), rather than Eq. 4.63 (with Q constant), in computing the force. One then obtains

$$F = -\frac{1}{2}V^2\frac{dC}{dx},$$

which is off by a sign. It is, of course, *possible* to maintain the capacitor at a fixed potential, by connecting it up to a battery. But in that case the *battery also does work* as the dielectric moves; instead of Eq. 4.59, we now have

$$dW = F_{\text{me}}\,dx + V\,dQ, \tag{4.66}$$

where $V\,dQ$ is the work done by the battery. It follows that

$$F = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}, \tag{4.67}$$

the same as before (Eq. 4.64), with the *correct* sign.

Please understand: The force on the dielectric cannot possibly depend on whether you plan to hold Q constant or V constant—it is determined entirely by the distribution of charge, free and bound. It's simpler to *calculate* the force assuming constant Q, because then you don't have to worry about work done by the battery; but if you insist, it can be done correctly either way.

Notice that we were able to determine the force *without knowing anything about the fringing fields that are ultimately responsible for it!* Of course, it's built into the whole structure of electrostatics that $\nabla \times \mathbf{E} = \mathbf{0}$, and hence that the fringing fields must be present; we're not really getting something for nothing here—just cleverly exploiting the internal consistency of the theory. The energy stored in the fringing fields themselves (which was not accounted for in this derivation) stays constant, as the slab moves; what *does* change is the energy well *inside* the capacitor, where the field is nice and uniform.

Problem 4.28 Two long coaxial cylindrical metal tubes (inner radius a, outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e, mass density ρ). The inner one is maintained at potential V, and the outer one is grounded (Fig. 4.32). To what height (h) does the oil rise, in the space between the tubes?

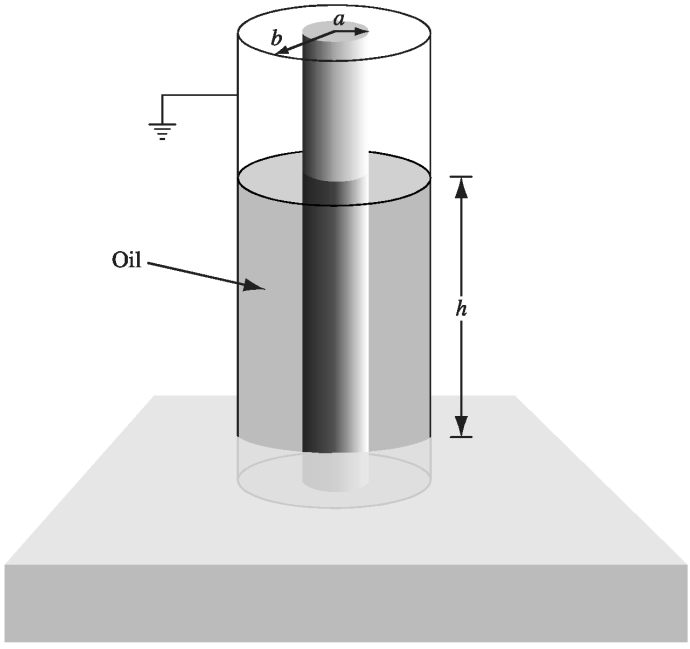


FIGURE 4.32

More Problems on Chapter 4

Problem 4.29

(a) For the configuration in Prob. 4.5, calculate the *force* on $\mathbf{p}_2$ due to $\mathbf{p}_1$, and the force on $\mathbf{p}_1$ due to $\mathbf{p}_2$. Are the answers consistent with Newton's third law?

(b) Find the total torque on $\mathbf{p}_2$ *with respect to the center of* $\mathbf{p}_1$, and compare it with the torque on $\mathbf{p}_1$ *about that same point*. [*Hint:* combine your answer to (a) with the result of Prob. 4.5.]

Problem 4.30 An electric dipole $\mathbf{p}$, pointing in the y direction, is placed midway between two large conducting plates, as shown in Fig. 4.33. Each plate makes a

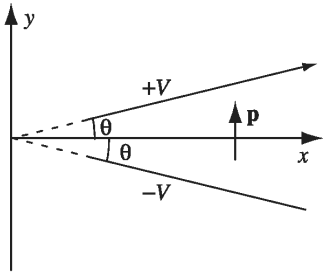


FIGURE 4.33

small angle θ with respect to the x axis, and they are maintained at potentials $\pm V$. What is the *direction* of the net force on **p**? (There's nothing to *calculate*, here, but do explain your answer qualitatively.)

Problem 4.31 A point charge Q is "nailed down" on a table. Around it, at radius R, is a frictionless circular track on which a dipole **p** rides, constrained always to point tangent to the circle. Use Eq. 4.5 to show that the electric force on the dipole is

$$\mathbf{F} = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}.$$

Notice that this force is always in the "forward" direction (you can easily confirm this by drawing a diagram showing the forces on the two ends of the dipole). Why isn't this a perpetual motion machine?[21]

! **Problem 4.32** Earnshaw's theorem (Prob. 3.2) says that you cannot trap a charged particle in an electrostatic field. *Question:* Could you trap a neutral (but polarizable) atom in an electrostatic field?

(a) Show that the force on the atom is $\mathbf{F} = \frac{1}{2}\alpha\nabla(E^2)$.

(b) The question becomes, therefore: Is it possible for E^2 to have a local maximum (in a charge-free region)? In that case the force would push the atom back to its equilibrium position. Show that the answer is *no*. [*Hint:* Use Prob. 3.4(a).][22]

Problem 4.33 A dielectric cube of side a, centered at the origin, carries a "frozen-in" polarization $\mathbf{P} = k\mathbf{r}$, where k is a constant. Find all the bound charges, and check that they add up to zero.

Problem 4.34 The space between the plates of a parallel-plate capacitor is filled with dielectric material whose dielectric constant varies linearly from 1 at the bottom plate ($x = 0$) to 2 at the top plate ($x = d$). The capacitor is connected to a battery of voltage V. Find all the bound charge, and check that the total is zero.

Problem 4.35 A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?

Problem 4.36 At the interface between one linear dielectric and another, the electric field lines bend (see Fig. 4.34). Show that

$$\tan\theta_2/\tan\theta_1 = \epsilon_2/\epsilon_1, \tag{4.68}$$

assuming there is no *free* charge at the boundary. [*Comment:* Eq. 4.68 is reminiscent of Snell's law in optics. Would a convex "lens" of dielectric material tend to "focus," or "defocus," the electric field?]

[21] This charming paradox was suggested by K. Brownstein.

[22] Interestingly, it *can* be done with *oscillating* fields. See K. T. McDonald, *Am. J. Phys.* **68**, 486 (2000).

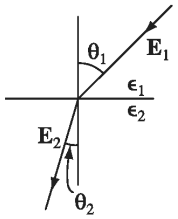


FIGURE 4.34

! **Problem 4.37** A point dipole $\mathbf{p}$ is imbedded at the center of a sphere of linear dielectric material (with radius R and dielectric constant ϵ_r). Find the electric potential inside and outside the sphere.

$$\left[\textit{Answer: } \frac{p\cos\theta}{4\pi\epsilon r^2}\left(1+2\frac{r^3}{R^3}\frac{(\epsilon_r-1)}{(\epsilon_r+2)}\right),\ (r\le R);\ \frac{p\cos\theta}{4\pi\epsilon_0 r^2}\left(\frac{3}{\epsilon_r+2}\right),\ (r\ge R)\right]$$

Problem 4.38 Prove the following uniqueness theorem: A volume $\mathcal{V}$ contains a specified free charge distribution, and various pieces of linear dielectric material, with the susceptibility of each one given. If the potential is specified on the boundaries $\mathcal{S}$ of $\mathcal{V}$ ($V = 0$ at infinity would be suitable) then the potential throughout $\mathcal{V}$ is uniquely determined. [*Hint:* Integrate $\boldsymbol{\nabla}\cdot(V_3\mathbf{D}_3)$ over $\mathcal{V}$.]

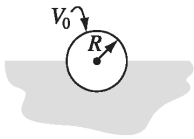


FIGURE 4.35

Problem 4.39 A conducting sphere at potential V_0 is half embedded in linear dielectric material of susceptibility χ_e, which occupies the region $z < 0$ (Fig. 4.35). *Claim:* the potential everywhere is exactly the same as it would have been in the absence of the dielectric! Check this claim, as follows:

(a) Write down the formula for the proposed potential $V(r)$, in terms of V_0, R, and r. Use it to determine the field, the polarization, the bound charge, and the free charge distribution on the sphere.

(b) Show that the resulting charge configuration would indeed produce the potential $V(r)$.

(c) Appeal to the uniqueness theorem in Prob. 4.38 to complete the argument.

(d) Could you solve the configurations in Fig. 4.36 with the same potential? If not, explain *why*.

FIGURE 4.36

Problem 4.40 According to Eq. 4.5, the force on a single dipole is $(\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}$, so the *net* force on a dielectric object is

$$\mathbf{F} = \int (\mathbf{P} \cdot \boldsymbol{\nabla})\mathbf{E}_{\text{ext}}\, d\tau. \tag{4.69}$$

[Here $\mathbf{E}_{\text{ext}}$ is the field of everything *except* the dielectric. You might assume that it wouldn't matter if you used the *total* field; after all, the dielectric can't exert a force on *itself.* However, because the field of the dielectric is discontinuous at the location of any bound surface charge, the derivative introduces a spurious delta function, and it is safest to stick with $\mathbf{E}_{\text{ext}}$.] Use Eq. 4.69 to determine the force on a tiny sphere, of radius R, composed of linear dielectric material of susceptibility χ_e, which is situated a distance s from a fine wire carrying a uniform line charge λ.

! **Problem 4.41** In a linear dielectric, the polarization is proportional to the field: $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$. If the material consists of atoms (or nonpolar molecules), the induced dipole moment of each one is likewise proportional to the field $\mathbf{p} = \alpha \mathbf{E}$. *Question:* What is the relation between the atomic polarizability α and the susceptibility χ_e?

Since $\mathbf{P}$ (the dipole moment per unit volume) is $\mathbf{p}$ (the dipole moment per atom) times N (the number of atoms per unit volume), $\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E}$, one's first inclination is to say that

$$\chi_e = \frac{N\alpha}{\epsilon_0}. \tag{4.70}$$

And in fact this is not far off, if the density is low. But closer inspection reveals a subtle problem, for the field $\mathbf{E}$ in Eq. 4.30 is the *total macroscopic* field in the medium, whereas the field in Eq. 4.1 is due to everything *except* the particular atom under consideration (polarizability was defined for an isolated atom subject to a specified external field); call this field $\mathbf{E}_{\text{elsc}}$. Imagine that the space allotted to each atom is a sphere of radius R, and show that

$$\mathbf{E} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\mathbf{E}_{\text{elsc}}. \tag{4.71}$$

Use this to conclude that

$$\chi_e = \frac{N\alpha/\epsilon_0}{1 - N\alpha/3\epsilon_0},$$

or

$$\alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right). \tag{4.72}$$

Equation 4.72 is known as the **Clausius-Mossotti** formula, or, in its application to optics, the **Lorentz-Lorenz** equation.

Problem 4.42 Check the Clausius-Mossotti relation (Eq. 4.72) for the gases listed in Table 4.1. (Dielectric constants are given in Table 4.2.) (The densities here are so small that Eqs. 4.70 and 4.72 are indistinguishable. For experimental data that confirm the Clausius-Mossotti correction term see, for instance, the first edition of Purcell's *Electricity and Magnetism,* Problem 9.28.)[23]

! **Problem 4.43** The Clausius-Mossotti equation (Prob. 4.41) tells you how to calculate the susceptibility of a *nonpolar* substance, in terms of the atomic polarizability α. The **Langevin equation** tells you how to calculate the susceptibility of a *polar* substance, in terms of the permanent molecular dipole moment p. Here's how it goes:

(a) The energy of a dipole in an external field $\mathbf{E}$ is $u = -\mathbf{p} \cdot \mathbf{E} = -pE\cos\theta$ (Eq. 4.6), where θ is the usual polar angle, if we orient the z axis along $\mathbf{E}$. Statistical mechanics says that for a material in equilibrium at absolute temperature T, the probability of a given molecule having energy u is proportional to the Boltzmann factor,

$$\exp(-u/kT).$$

The average energy of the dipoles is therefore

$$<u> = \frac{\int u e^{-(u/kT)}\, d\Omega}{\int e^{-(u/kT)}\, d\Omega},$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$, and the integration is over all orientations ($\theta: 0 \to \pi$; $\phi: 0 \to 2\pi$). Use this to show that the polarization of a substance containing N molecules per unit volume is

$$P = Np[\coth(pE/kT) - (kT/pE)]. \tag{4.73}$$

That's the Langevin formula. Sketch P/Np as a function of pE/kT.

(b) Notice that for large fields/low temperatures, virtually *all* the molecules are lined up, and the material is *non*linear. Ordinarily, however, kT is much greater than pE. Show that in this régime the material *is* linear, and calculate its susceptibility, in terms of N, p, T, and k. Compute the susceptibility of water at 20°C, and compare the experimental value in Table 4.2. (The dipole moment of water is 6.1×10^{-30} C·m.) This is rather far off, because we have again neglected the distinction between $\mathbf{E}$ and $\mathbf{E}_{\text{clsc}}$. The agreement is better in low-density gases, for which the difference between $\mathbf{E}$ and $\mathbf{E}_{\text{else}}$ is negligible. Try it for water vapor at 100°C and 1 atm.

---

[23] E. M. Purcell, *Electricity and Magnetism* (Berkeley Physics Course, Vol. 2), (New York: McGraw-Hill, 1963).