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Mathematical method In physics

PHY 307

Topic: VECTOR SPACE

Definition: A Vector is any quantity that has both magnitude and direction It is denoted by either $\underline{a}$ or $\bar{a}$ or $a$

A field is any set F on which is defined as a pair of binary operation denoted by + and · called addition and multiplication respectively such that

① $\alpha+\beta = \beta+\alpha$; $\forall\ \alpha, \beta \in F$, ($\forall$ = for all)
② $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$; $\forall\ \alpha, \beta, \gamma \in F$
③ $\exists\ 0 \in F$ called zero; $\alpha+0=\alpha$, $\forall\ \alpha \in F$
④ $\forall\ \alpha \in F$ $\exists\ -\alpha \in F$ called additive inverse of $\alpha$; $\alpha+(-\alpha)=0$ $\forall\ \alpha \in F$

⑤ $\beta\cdot\alpha = \alpha\cdot\beta$ $\forall\ \alpha, \beta \in F$

⑥ $(\alpha\cdot\beta)\cdot\gamma = \alpha\cdot(\beta\cdot\gamma)$

⑦ $\exists\ 1 \in F$ called unit : $\alpha\cdot 1 = \alpha$

⑧ $\forall\ \alpha \in F$ and $\alpha \neq 0$ $\exists\ \alpha^{-1}$ called the multiplicative Inverse or reciprocal of $\alpha$; $\alpha\cdot\alpha^{-1}=1$

⑨ $\alpha\cdot(\beta+\gamma) = \alpha\cdot\beta+\alpha\cdot\gamma$ $\forall\ \alpha, \beta, \gamma \in F$

## Definition

The elements of the field are called scalars.

Examples Includes

* The set of all real numbers are together with the

usual operations + and • (addition and multiplication is a field)

2 The set of all rational numbers $Q$ together with the usual operations of addition and multiplication is a field.

3 The set of all Complex numbers $\mathbb{C}$ together with the usual operations of + and • is a field.

## Definition

Let $V$ be the set on which is defined, a binary operation + called addition such that:

① $a+b = b+a, \ \forall a,b \in V$

② $(a+b)+c = a+(b+c), \ \forall a,b,c \in V$

③ $\exists 0 \in V$ called zero ; $a+0 = a \ \forall a, \in V$

④ $\forall a \in V \ \exists -a \in V$ called additive Inverse or $-a : a+(-a) = 0$

Let $F$ be a field and • the binary operation which associates with $\alpha \in F$ and a $a \in V$ and $\alpha \cdot a$ of $V$:

⑤ $\alpha \cdot (\beta \cdot a) = (\alpha \cdot \beta) \cdot a \ ; \forall \alpha, \beta \in F ; \ a \in V$

⑥ $\exists 1 \in F$ (unit): $1 \cdot a = a \ ; \ \forall a \in V$

⑦ $\alpha \cdot (a+b) = \alpha \cdot a + \alpha \cdot b \ ; \ \forall \alpha \cdot \forall F \ a,b \in V$

⑧ $(\alpha + \beta) \cdot a = \alpha \cdot a + \beta \cdot a, \ \forall \alpha, \beta \in F ; a \in V$ then $V$ is said to be a Vector space over the field $F$

Definition

· The element of the Vector space are called Vectors.
The Vector $\alpha \cdot a$ where $\alpha \in F$ and $a \in V$ is called the multiple of $a$ and $\alpha$

Examples

Let $R^1$ be the set of all real numbers, Let + be the set of usual operation of ~~rela~~ real numbers. ~~Let~~
Let · be the usual operation of multiplication of real numbers then $R^1$ is a ~~Vecto~~ Vector space over the real field $R$

Examples

Let $R^2$ be the set of all pairs of real numbers of the form form $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Let + be the operation of addition defined on $R^2$ such that If we have $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then

$$a + b = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

Let · be the operation of multiplication defined by

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \lambda \in R$$

$$\lambda \cdot a = \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \end{pmatrix}$$ then $R^2$ is a Vector space over the real field $R$ which is called the real Plain → (explained)

## Examples:

Let $R^3$ be the set of all triats of real numbers of the form $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

Let $+$ be the operation of addition defined on $R^3$, then,

$$a+b = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

Similarly, let $\cdot$ be the operation of multiplication defined on $R^3$ then, $\lambda \in R$

$$\lambda . a = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{pmatrix}, \quad \lambda . b = \begin{pmatrix} \lambda b_1 \\ \lambda b_2 \\ \lambda b_3 \end{pmatrix}$$

Then $R^3$ is a vector space over the real field $R$ which is called the <u>real space</u>

### Example

Let $n$ be any natural number greater than 3, let $R^n$ be the set of all ordered $n$ - tripples of real numbers of the form $a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

Let $+$ and $\cdot$ be the usual operation of addition & multiplication respectively on $R^n$ then,

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad a+b = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n \quad b_n \end{pmatrix}$$

⑤ $\forall$ - for all

also

$$\lambda \cdot a = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}$$ then $R^n$ is a vector space over the real space $R$

Example ①

$x_1 = (1, 2, 3)$
$x_2 = (3, -1, 4)$
$x_3 = (4, 1, -7)$

for a vector to be linearly independent

$$\nexists \sum_{i=1}^{n} \alpha_i x_i = 0 \Rightarrow \alpha_i \neq 0 \ \forall i$$

$$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$$

On substitution

$$\alpha_1 (1, 2, 3) + \alpha_2 (3, -1, 4) + \alpha_3 (4, 1, -7) = 000$$

$$(\alpha_1, 2\alpha_1, 3\alpha_1) + (3\alpha_2, -\alpha_2, 4\alpha_2) + (4\alpha_3, \alpha_3, -7\alpha_3 \ldots$$

collecting the corresponding terms.

$\alpha_1 + 3\alpha_2 + 4\alpha_3 = 0$ ——— ①
$2\alpha_1 - \alpha_2 + \alpha_3 = 0$ ——— ②
$3\alpha_1 + 4\alpha_2 - 7\alpha_3 = 0$ ——— ③

solve for eqn ①, ② and 3 simultaneously

$\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -1$

The conclusion is that the vectors are linearly independent because the $\alpha$ are not zero.

$\alpha_1 + 3\alpha_2 + 4\alpha_3 = 0$ —①

$2\alpha_1 - \alpha_2 + \alpha_3 = 0$ —②

$3\alpha_1 + 4\alpha_2 - 7\alpha_3 = 0$ —③

Solution

putting from eqtn ①

$\alpha_1 = -3\alpha_2 + 4\alpha_3$ —④

putting 4 into ②

$2(-3\alpha_2 - 4\alpha_3) - \alpha_2 + \alpha_3 = 0$

$-6\alpha_2 - 8\alpha_3 - \alpha_2 + \alpha_3 = 0$

like terms

$-6\alpha_2 - \alpha_2 - 8\alpha_3 + \alpha_3 = 0$

$-7\alpha_2 - 7\alpha_3 = 0$

$-7\alpha_2 = 7\alpha_3$

$\alpha_2 = -\alpha_3$

~~$\alpha_2 =$~~ $\therefore \alpha_3 = -1$

$\alpha_1 = 1, \alpha_2 = 1$

putting $\alpha_3$ and $\alpha_2$ in eqtn ①

to check

$\alpha_1 + 3(1) + 4(-1) = 0$

$\alpha_1 + 3 - 4 = 0$

$\alpha_1 - 1 = 0$

$\therefore \alpha_1 = 1$ //

putting $\alpha_1, \alpha_2$ into eqtn

$2(1) - (1) + \alpha_3 = 0$

$2 - 1 + \alpha_3 = 0$

$1 + \alpha_3 = 0$

$\alpha_3 = -1$ ///

putting $\alpha_1$ and $\alpha_2$ in ①

~~$(+1) + 3(-1)$~~

$① + 3\alpha_2 - 4(1) = 0$

$1 + 3\alpha_2 - 4 = 0$

$3\alpha_2 - 3 = 0$

$\frac{3\alpha_2}{3} = \frac{3}{3}$

$\alpha_2 = \underline{\underline{1}}$

Example $\boxed{2}$: Are the vectors

$X_1 = (1, 0, 0)$

$X_2 = (0, 1, 0)$

$X_3 = (0, 0, 1)$ linearly dependent?

Solution

for vectors to be linearly dependent

1b② $\sum_{i=1}^{n} \alpha_i x_i = 0 \Rightarrow \alpha_i \neq 0 \; \forall_i$

Expand

$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$

$\alpha_1(1, 0, 0) + \alpha_2 (0, 1, 0) + \alpha_3 (0, 0, 1) = 0$

$(\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0, 0, \alpha_3) = 0$

Collect the corresponding terms

$\alpha_1 + 0 + 0 = 0$ —①

$0 + \alpha_2 + 0 = 0$ —②

$0 + 0 + \alpha_3 = 0$ —③

solving simultaneously, we have

$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$ i.e $\alpha_1 = \alpha_2 = \alpha_3 = 0$

In conclusion, since all the $\alpha$ are zero then the $\alpha$ is linearly Independent

Supposing some $\alpha$ are zero (0) and some are not then, they are linearly dependent.

⑦

## Simultaneous eqtn ⑤

$\alpha_1 + 0 + 0 = 0$ — ①

~~0~~ $+ \alpha_2 + 0 = 0$ — ②

$0 + 0 + \alpha_3 = 0$ — ⑤

from eqtn ① $\alpha_1 + 0 + 0 = 0$

$\alpha_1 = 0$

From eqtn ② $0 + \alpha_2 + 0 = 0$

$\alpha_2 = 0 + 0$

$\alpha_2 = 0$

~~from~~

from eqtn ③

$\alpha_3 = 0$

It is linearly Independent.

from pg ⑨

Make $\alpha_2$ the subject formula in eqtn ②

$\alpha_2 = -6\alpha_3$ — ⑤

Also make $\alpha_1$ the subject formula in eqtn ①

$\alpha_1 = 6\alpha_4 - 4\alpha_3 - 3\alpha_2$ — ⑥

Subst. $\alpha_1$ and $\alpha_2$ into eqtn ③

$2(6\alpha_4 - 4\alpha_3 - 3\alpha_2) + 2(-6\alpha_3) + 2\alpha_3 - 3\alpha_4 = 0$

$12\alpha_4 - 8\alpha_3 - 6\alpha_2 - 12\alpha_1 + 2\alpha_3 - 3\alpha_4 = 0$

~~opp~~ like terms

$12\alpha_4 - 3\alpha_4 - 8\alpha_3 - 12\alpha_3 + 2\alpha_3 - 6\alpha_2 = 0$

$9\alpha_4 - 18\alpha_3 - 6\alpha_2 = 0$

but $\alpha_2 = -6\alpha_3$

$\therefore 9\alpha_4 - 18\alpha_3 - 6(-6\alpha_3) \Rightarrow 9\alpha_4 - 18\alpha_3 + 36\alpha_3$

$9\alpha_4 + 12\alpha_3 = 0$

$9\alpha_4 = -12\alpha_3 \Rightarrow \frac{9\alpha_4}{9} = -\frac{12\alpha_3}{9}$

$\alpha_4 = -2\alpha_3$

$\therefore \alpha_4 = -2 \; ; \; \alpha_3 = 1$

Subst. $\alpha_3$ into eqtn ⑤

$\alpha_2 = -6(1) = -6$

Subst $\alpha_4, \alpha_3$ and $\alpha_2$ into eqtn ⑥

$\alpha_1 = 6(-2) - 4(1) - 3(6) = -12 + 4 + 18 = 2$

$\therefore \alpha_1 = 2, \alpha_2 = -6, \alpha_3 = 1$ and $\alpha_4 = -2$

Therefore, we conclude that, the Vectors are linearly dependent.

Example: Show that whether or not the Vectors ~~are~~

$X_1 = (1, 0, 2, 1)$

$X_2 = (3, 1, 2, 1)$

$X_3 = (4, 6, 2, -4)$

$X_4 = (-6, 0, -3, -4)$ are linearly dependent

Solution

For Vectors to be linearly dependent

$\sum_{i=1}^{n} \alpha_i X_i = 0 \Rightarrow$ ~~the~~ $\alpha_i \neq 0 \forall i$

Expanding

$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 = 0$

On Substitution

$\alpha_1(1, 0, 2, 1) + \alpha_2(3, 1, 2, 1) + \alpha_3(4, 6, 2, -4) + \alpha_4(6, 0, -3, -4) = 0$

$$(\alpha_1, 0, 2\alpha_1, \alpha_1) + (3\alpha_2, \alpha_2, \cancel{\alpha_2}\ 2\alpha_2, \alpha_2) + (4\alpha_3, 6\alpha_3, 2\alpha_3, -4\alpha_3) + (-6\alpha_4, 0, -3\alpha_4, -4\alpha_4) = 0000$$

Collect for corresponding terms

$$\left.\begin{array}{ll} \alpha_1 + 3\alpha_2 + 4\alpha_3 - 6\alpha_4 = 0 & \text{——} ① \\ 0 + \alpha_2 + 6\alpha_3 + 0 = 0 & \text{——} ② \\ 2\alpha_1 + 2\alpha_2 + 7\alpha_3 - 3\alpha_4 = 0 & \text{——} ③ \\ \alpha_1 + \alpha_2 + 4\alpha_3 - 4\alpha_4 = 0 & \text{——} ④ \end{array}\right\} \text{——} (*)$$

$\alpha_1$

Conclusion

~~Definition~~ Definition

The number of element in a basis of a vector space is called its dimension

Definition

The vector ~~s~~space is said to be finite dimensional, if it has a finite basis otherwise, it is said to be infinite dimensional

Example: The subset

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ of } R^3$$

is linearly or are linearly independent and for every vector ie $$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in R^3$$

$$a = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Consequently, the subset is a basis for $R^3$. Thus $R^3$ has dimension 3

- Theorem: All bases of a Vector space has the same dimension

- Theorem: ~~Every~~ linearly ~~ind~~ Independent set of Vectors from a Vector space may be extended to the basis for that Vector. The [illegible] of this Vector is called Gramm - Schmidt method

- Theorem: Every subset of n+1 Vectors from n-dimensional Vector space is linearly dependent

## AN OPERATOR

A rule (mapping or correspondence) A which associate with each Vector $x$ in a Vector space V, a unique Vector denoted by $Ax$ in V is called an operator on the Vector space V

### Definition

An operator A on the Vector space V is said to be linear if ∀ ~~all~~ $\forall\, x, y \in V$ and $\lambda \in F$ (field)

① $A(x+y) = Ax + Ay$
② $A(\lambda x) = \lambda Ax$

### Example

Let A be Vector and $R^2$ defined such that $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$ then

$\cdot) \to$ then [illegible]

$$Ax = \begin{pmatrix} x_1 \cos\alpha - x_2 \sin\alpha \\ x_1 \sin\alpha + x_2 \cos\alpha \end{pmatrix}$$ for a fixed constant $\alpha$.

~~then, if~~

then if $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in R^2$, it follows that $A(x+y) = A\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$

Also $= A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$= Ax + Ay$

Similarly $A(\lambda x) = A\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$

$= \lambda A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= \lambda A x$

operator

Example 2 Let $A$ be a differentiation Vector defined on a Vector space

$A$, $C(a,b)$, $a, b \in R$

$Ay = \frac{d}{dx} y(x)$; $\forall y \in C(a,b)$, $x \in (a,b)$

then $y_1, y_2 \in C(a,b)$ it follows that

$A(y_1+y_2) = \frac{d}{dx}[y_1(x) + y_2(x)]$

$= \frac{d}{dx} y_1(x) + \frac{d}{dx} y_2(x)$

$= Ay_1 + Ay_2$

Similarly: if ~~$\alpha$ is an element~~

$\lambda \in R$ then

$A(\lambda y) = \frac{d}{dx}[\lambda y(x)] = \lambda \frac{d}{dx}[y(x)]$

$= \lambda A y$

We can say consequently that A is linear.

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Let A be a differential operator defined in ~~operator~~ vector space $C(a,b)$, $a,b \in R$ by

$$Ay = \int y(x)dx, \forall y \in C(a,b), x \in (a,b)$$

then for $y_1, y_2 \in C(a,b)$

it follows that $A(y_1+y_2) = \int [y_1(x)+y_2(x)]dx$

Again $= \int y_1(x)dx + \int y_2(x)dx$

$= Ay_1 + Ay_2$

It is a linear operator because it agrees

Also if $\lambda \in R$, then

$$A(\lambda y) = \int [\lambda y(x)]dx$$
$$= \lambda \int y(x)dx$$
$$= \lambda A y(x)$$

~~And~~ This indicates that +, · and × are linear operator.

**Definition:** If A, B are two linear operator in a vector space V, then their sum denoted by $(A+B)$ then it is said that ~~(at~~ $Ay+By$, $\forall y \in V$ Therefore summation is a linear operator.

Similarly, the product denoted by A, B is defined by

$$(A \cdot B)y = A(By), \forall y \in V$$

**Definition:** The linear operator O defined on the vector space V by $(A+O)y = Ay$, for all linear operator A on V and all $y \in V$ is the zero operator on V

**Definition**: The linear operator I defined on a vector space V by y ; $\forall y \in V$, is called a unit operator on V
i.e $Iy = y ; \forall y \in V$.

**Definition**: If A is a non-zero linear operator on a vector space V, then the operator denoted by $A^{-1}$ on the V such that

$$(A^{-1}A)y = (AA^{-1})y$$
$$= Iy$$
$$= y ; \forall y \in V$$ is called an Inverse Operator A

**Definition**: ~~Let~~ If A be a linear operator on a vector
If a linear operator has an Inverse ~~is~~ it is said to be Invertable

**Theorem**: If A and B are Invertable linear operators, then A

$$(AB)^{-1} = B^{-1}A^{-1}$$ and

$$(A^{-1})^{-1} = A$$

**Definition**: Let A be linear operator on vector over a field F.
If $y(\neq 0) \in V$ and
$\lambda(\neq 0) \in F$

Note that

$$Ay = \lambda y \longrightarrow *$$

then $\lambda$ and y are called eigen vector and each corresponds to eigen Value, of ~~respectively~~ respectively of the operator A. equation ⊛ above is the eigen equation or eigen problem for the operator A

**Definition**

Let V be a vector space over the Complex space field C there a binary operation on V denoted by $\langle | \rangle$, is said

$\langle | \rangle$ - Inner Product.

$\overline{\langle z|y \rangle}$ = conjugate.

is said to be an Inner product on V; if $\forall y, z \in V$,
① $\forall y, z \in V$ ② $\langle y|z \rangle \in C$ and
③ $\forall \lambda, \mu \in \mathbb{C}$ ④ $w \in V$
the following condition holds.
ⓐ $\langle \lambda y + \mu z | w \rangle = \lambda \langle y|w \rangle + \mu \langle z|w \rangle$
ⓑ $\langle y|z \rangle = \overline{\langle z|y \rangle}$
ⓒ $\langle y|y \rangle \geq 0$
ⓓ $\langle y|y \rangle = 0 \iff y = 0$

Example: let $\langle | \rangle$ be the ~~Inner product~~ binary operation defined on $R^3$ given by.

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$ then the Inner product of

$\langle y|z \rangle = y_1 z_1 + y_2 z_2 + y_3 z_3$ then

$\langle | \rangle$ is an Inner product on $R^3$

Example let $\langle | \rangle$ be the binary operation defined on the space $C(a,b)$ of all piece wise, continuous and smooth function over the real interval $(a,b)$ such that, if ~~$y, z \in C(a$~~ $y, z \in C(a,b)$, then the ~~$\langle z \rangle = \int_a^b y$~~

~~$\langle z \rangle =$~~ $$\boxed{\langle y|z \rangle = \int_a^b \overline{y(x)}\, z(x)\,dx}$$

then the $\langle y|z \rangle$ is an Inner product on $C(a,b)$

**Definition**, the vector space on which is defined on an Inner product is called an Inner product space and is denoted by $\{V, \langle | \rangle\}$

**Theorem**: If $y, z$ is are elements of an Inner product space and $\lambda$ is a scalar then the $\langle y | \lambda z \rangle = \overline{\lambda} \langle y|z \rangle$
$= \overline{\lambda}(y_1 z_1 + y_2 z_2 + ---)$

$\sqrt{\langle y|y\rangle} = \sqrt{y_1 y_1 + y_2 y_2 + \cdots}$

Definition:

If $y$ is an element of an inner product space, then the real number $\sqrt{\langle y|y\rangle}$ is called the norm of $y$ and denoted by $\|y\|$ = is the same as $\sqrt{\langle y|y\rangle}$

Definition: A vector space ~~V~~ V on which is defined a norm is called a normed vector space and denoted by

$\{V, \|\cdot\|\}$

Theorem: If $y$ is an element of a ~~non~~ $\overset{\text{normed}}{\wedge}$ vector space, then

$y \neq 0 \Longleftrightarrow \|y\| > 0$

Theorem: If $y$ is an element of a normed vector space and $\lambda \in C$, then

~~H.C~~ $\|\lambda y\| = |\lambda| \|y\|$

Note $|\lambda| \|y\|$

$= |\lambda| \sqrt{\langle y|y\rangle}$

$= |\lambda| \sqrt{y_1 y_1 + y_2 y_2 + \cdots}$

$= |\lambda| \int_a^b \bar{y}(x) y(x) dx$

Definition: Two vectors $y, z$ of an inner product space are said to be orthogonal if the inner

$\langle y|z\rangle = 0$ ✱ Note

Definition: A vector $y$ of ~~nveme~~ normed vector space, is said to be normalised ~~is said~~ if

$\|y\| = 1$

as $\|y\| = \sqrt{\langle y|y\rangle}$

**Definition**: A normed vector space is called a Hilbert space if ~~fo~~ ~~$\forall y \in V$~~ $\forall y, z \in V$ and $\lambda \in C$

$\exists$ = They exist

① $\|\lambda y\| = |\lambda| \, \|y\|$

② $\|y\| = 0 \Longleftrightarrow y = 0$

③ $\|y+z\| \leq \|y\| + \|z\|$

④ If $\{y_n\}$, is a sequence of element $V$ such that $\|y_m - y_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$ then $\exists \, y \in V$; $\|y_n - y\| \longrightarrow$ as $n \longrightarrow \infty$

Assignment

**Definition:** the Inequality $\|y+z\| \leq \|y\| + \|z\|$ is called the triangular or Schwarz's Inequality

**Prep:** Proof the triangular Inequality

**Definition:** If a sequence $\{y_n\}$ of element of a normed Vector space satisfying the condition that if $\|y_n - y\| \to 0$ as $n \to 0$ is called a Cauchy Sequence.

**Definition:** A sequence $\{y_m\}$ of element of a normed Vector space $\{V, \|\cdot\|\}$ is said to converge to $y \in V$ if $\|y_m - y\| \to 0$ as $m \to \infty$

**Definition:** A sequence $\{y_m\}$ of element of a normed Vector space satisfying the condition that $\|y_m - y_n\| \to 0$, as $m, n \to \infty$, $\|y_n - y\| \to 0$, $n \to \infty$ ~~is said to be complete~~ is said to be Complete

**Definition:** A sequence $\{y_n\}$ of element of ~~a base Vect~~ normed Vector space is said to be orthogonal, if the inner product of $\langle y_n / y_m \rangle = 0$; $m \neq n$

## Solution to Assignment

To proof the Magnitude or Schwarz's Inequality
~~It can be :~~
It can be shown that $\|y + z\| \leq \|y\| + \|z\|$ —— ①

~~$\|y+z\|^2 = \sum_{i=1}^{n}$~~ $\|y+z\| \leq \|y\| + \|z\|$ —— ①

Square the LHS of the eqtn ① and Cauchy Schwarz's Inequality ~~at~~

$$\|y+z\|^2 = \sum_{i=1}^{n} (y_i + z_i)^2 \text{ —— } ②$$

$$\|y+z\|^2 = (y_1 + z_1)^2 + (y_2 + z_2)^2 + \cdots (y_n + z_n)^2$$

Simplify it

$$\|y+z\|^2 = (y_1^2 + 2y_1 z_1 + z_1^2) + (y_2^2 + 2y_2 z_2 + z_2^2) + \cdots (y_n^2 + 2y_n z_n + z_n^2)$$

~~$\|y+z\|^2 = (y_1^2 + y_n^2$~~ collect the corresponding terms

$$\|y+z\|^2 = (y_1^2 + y_2^2 + \cdots y_n^2) + 2(y_1 z_1 + y_2 z_2 + \cdots y_n z_n) + (z_1^2 + z_2^2 + \cdots z_n^2)$$

$$\|y+z\|^2 = \|y\|^2 + 2(y \cdot z) + \|z\|^2$$

Now Apply the Inequality

$$\|y+z\|^2 \leq \|y\|^2 + 2\|y\|\|z\| + \|z\|^2$$

$$\|y+z\|^2 \leq (\|y\| + \|z\|)^2 \text{ —— } ②$$

Taking the square root of ~~eqtn ②~~ both sides of eqtn ②

$$\sqrt{\|y+z\|^2} \leq \sqrt{(\|y\| + \|z\|)^2}$$

$\|y+z\| \leq \|y\| + \|z\|$ ——— ③ ///

eqn ③ Complete the proof, since it satisfyes eqn ①

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A sequence $y_n$ of ~~an~~ element of an Inner product space is said to be orthonormal if

$$\langle y_n | y_m \rangle = \delta_{m,n}$$

where ~~$\delta_{mn}$ is~~ $\delta_{mn}$ is the Kronecker delta

**Theorem:**

If $\{y_n\}$ is a complete orthonormal Sequence from a normal Vector space and ~~y is any in that~~ $y \in V$ then,

$$\boxed{y = \sum_n \langle y | y_n \rangle y_n} \text{———} ④$$

which is called the fourier Series expansion of y wrt a sequence $\langle y_n \rangle$

**Definition:** The number $\langle y | y_n \rangle$ In eqn ④ are called the fourier Coefficient of y w.r.t the Sequence ~~$\langle y_n \rangle$~~ $\{y_n\}$

**Theorem**

If the inner product of $\langle y | y_n \rangle$ where the fourier Coefficient of y w.r.t the sequence is $\{y_n\}$ and the Sequence $\{y_n\}$ is orthonormalised, then

$$\| y \|^2 = \sum_n | \langle y | y_n \rangle |^2$$

**Definition:** A linear operator A on an inner product space is said to be symmetric if $\forall y, z \in V$

$$\langle Ay | z \rangle = \langle y | Az \rangle$$

**Theorem:** The eigen Values of a symmetric operator are real

**Theorem:** If A is a Symmetric Operator on an Inner product

1. + 0

Space and $y, z \in V$ are eigen Vector of A Corresponding to different eigen Values, then

$$\langle y | z \rangle = 0$$

**Theorem:** If X and y are two Vector in real $R^n$, then

~~$x \cdot y =$~~

$$x \cdot y = \frac{1}{4} \|x+y\|^2 - \frac{1}{4} \|x-y\|^2$$

This is the relationship between the inner product and the norm -

**Proof**

$$\|x+y\|^2 = \|x\|^2 + 2(x \cdot y) + \|y\|^2 \text{ ———— } ①$$

$$\|x-y\|^2 = \|x\|^2 - 2(x \cdot y) + \|y\|^2 \text{ ———— } ②$$

Substracting eqtn ② from ①

$$4(x \cdot y) = \|x+y\|^2 - \|x-y\|^2$$

Divide all through by 4

$$x \cdot y = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

**Definition:** Let $R^3$ be the set of all triples of real numbers of the form

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Let + be the usual operation of $\overset{\text{Addition}}{\text{multiplication}}$ defined on $R^3$ such that ~~$\lambda \in R$, then~~ if $x, y \in R^3$ are given by

$$\cancel{\lambda} x = \cancel{\lambda} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \overset{\text{and}}{\cancel{=}} \cancel{\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix}} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

then $x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$

Also,
Let • be the Operation of multiplication defined on $R^3$ such that $\lambda \in R$, then

$$\lambda \cdot x = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix}$$

then, it follows ~~that~~ trivially that

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3$$

which is a linear combination of all elements of the subset. Consequently, the subset is a basis for $R^3$, thus the dimension of $R^3$ is 3

The inner product of two elements  05/09/2019

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$ may be defined as the

, i.e. $\langle x|y\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$ Consequently $R^3$ is an inner product space.

This inner product induces the norm defined by
Thus $\|x\| = \left(x_1^2 + x_2^2 + x_3^2\right)^{\frac{1}{2}}$ thus making $R^3$ a normed vector space

<u>Theorem:</u> If
If $x \in R^3$ and $\hat{x}$ is the vector defined by $\frac{1}{\|x\|}x$,
i.e. $\hat{x} = \overset{\text{unit vector}}{\frac{1}{\|x\|}x}$, then

$\|\hat{x}\| = 1$ and this is a unit vector. In this case $\hat{x}$ is said to be normalised

# Eigen Value and Eigen Vector

Let $x \in R^3$ be an eigen Vector of any matrix operator A Corresponding to the eigen Value $\lambda$, then I Clai Claime that $Ax = \lambda x$ —— ①

It follows that the linear equation for the eigen Values is given by determinantal equation.

$$|A - \lambda I_3| = 0 \quad \text{—— ②}$$

~~$I_3$ as three by three matrix~~ where $I_3$ is a unit three by three matrix. This determinantal eqn ② is called characteristics eqn. Each eigen Value then determine ~~and~~ it eigenVector which may be normallised.

The eqn ② may be written generally as

$$|A - \lambda I| x = 0 \quad \text{—— ③}$$

where $\lambda$ is the eigen Value of the matrix A and $x$ is the corresponding non-zero eigen Vector.

Example:

Let $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ find the Characteristics matrix and the Corresponding eigen Values.

Solution

The characteristic eqtn is

$$|A - \lambda I_3| = 0$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|A-\lambda I_3| = \left| \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$ matrix eqn

$$(2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2[1(2-\lambda)-(1)(1)] + 1[1(2)-(3-\lambda)(1)] = 0$$ ~~xtic eqn~~

Solving the above gives

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$ xtics eqn

$$(\lambda-1)(\lambda-1)(\lambda-5) = 0$$

~~$\lambda_1 =$~~ solving the above yields

$\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 5$

~~$(2-\lambda)[(3-\lambda)(2-\lambda)(1)2] - 2[1(2-\lambda)-(1)(1)] + 1[1(2)-(3-\lambda)(1)]$~~

complete solution to the above problem

$|A - \lambda I_3| = 0$

$$\begin{vmatrix} \overset{+}{2-\lambda} & \overset{-}{2} & \overset{+}{1} \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$2-\lambda \begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 3-\lambda \\ 1 & 2 \end{vmatrix} = 0$$

$$2-\lambda[(3-\lambda)(2-\lambda)-2(1)] - 2[(1)(2-\lambda)-(1)(1)] + 1[(1)(2)-(1)(3-\lambda)] = 0$$

$$2-\lambda(6-3\lambda-2\lambda+\lambda^2-2] - 2(2-\lambda-1) + 1(2-3+\lambda) = 0$$

$$2-\lambda(4-5\lambda+\lambda^2) - 2(1-\lambda) + 1(-1+\lambda) = 0$$

$$8 - 10\lambda + 2\lambda^2 - 4\lambda + 5\lambda^2 - \lambda^3 - 2 + 2\lambda - 1 + \lambda = 0$$

collect like terms

$$-\lambda^3 + 2\lambda^2 + 5\lambda^2 - 10\lambda - 4\lambda + 2\lambda + \lambda + 8 - 2 - 1 = 0$$

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

multiply − through, i.e hence,

$$\lambda^3 - 7\lambda^2 + 11\lambda + 5 = 0$$

Testing for the factor, $\lambda - 1 = 0 \Rightarrow \lambda_1 = 1$

i.e

$$(1)^3 - 7(1)^2 + 11(1) - 5 = 0$$

$\therefore \lambda = 1$ is a factor, we now use long d.

$$\begin{array}{r} \lambda^2 - 6\lambda + 5 \\ \lambda - 1 \overline{\big) \lambda^3 - 7\lambda^2 + 11\lambda - 5} \\ -\ \lambda^3 - \lambda^2 \\ \hline -6\lambda^2 + 11\lambda - 5 \\ -\ -6\lambda^2 + 6\lambda \\ \hline 5\lambda - 5 \\ -\ 5\lambda - 5 \\ \hline 0 \quad 0 \end{array}$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \left( \begin{array}{l} \text{where } a = 1, b = -6 \\ c = 5 \end{array} \right)$$

$$\frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)} = \frac{6 \pm \sqrt{36 - 20}}{2}$$

$$\frac{6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2}$$

$$\therefore \frac{6+4}{2} \text{ or } \frac{6-4}{2}$$

$$\frac{10}{2} \text{ or } \frac{2}{2}$$

$5$ or $1$, $\lambda_1 = 1$

$\therefore \lambda_2 = 1$ and $\lambda_3 = 5$

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find the Eigen Value of

$$A = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$\left| \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right|$$

after solving we have

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$\lambda_1 = 2, \ \lambda_2 = 2, \ \lambda_3 = 8$

~~Example~~ Exercise ② find the Eigen Value of

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

After solving we have

$\lambda = 1, 1, 3$

# EIGEN VECTOR

A Column Vector X may be transformed into a column Vector y by using a Vector Matrix and multiplying the Column Vector X by a scalar quantity $\lambda$ so that we find the same transformed Column Vector y ie

$$AX = \lambda X \quad \text{——} ①$$

Here X is the eigen Vector

Example 1: Show that

X = (1, 1, 2) is an eg eigen Vector of the matrix

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$$

Sol<sup></sup>ⁿ since x = (1, 1, 2) and A is also known

$$\therefore \cancel{\text{let}}\ AX = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix} = \quad 3 \cancel{+1-2}$$

or

$$\cancel{Ax} = Ax = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3+1-2 \\ 2+2-2 \\ 2+2+0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \Rightarrow 2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 2X$$

$$\lambda = 2, \ [Ax = \lambda \mathrm{I}] \cancel{x = 0}$$

$$\begin{pmatrix} \vdots & \vdots & \vdots \end{pmatrix}$$

Each Eigen ~~λ~~ Value has a Corresponding $\overset{\text{non}}{\wedge}$ ~~Nomerf~~ zero ~~Vector~~ Vector x which satisfy eigen eqn $(A - \lambda I)X = 0$
The non-zero Vector X is called ~~X~~ latent Vector or Eigen Vector

## CHARACTERISTICS OR PROPERTIES OF EIGEN VECTORS

① The Eigen Vectors of a Matrix A is not <u>Unique</u>
② If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct eigen Values of an $m \times m$ matrix with the Corresponding eigen Vectors $x_1, x_2, \ldots, x_n$ form a <u>linearly Independent</u> set
③ If two or more eigen Values are equal, <u>it may or may not</u> be possible to get linearly Independent eigen Vectors corresponding with equal root
④ Two eigen Vectors are said to $\overset{\text{be}}{\wedge}$ Orthogonal if the eigen Vectors

$X_1 \longrightarrow \quad X_1' X_2 = 0$

explanation

$$X_1' X_2 = 0 \Rightarrow X_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ is same as } X_1' (1, 1, 2)$$

$$X_1' X_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$X_1' = (1 \quad 7 \quad 2) \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = 0 \ \therefore \text{It is orthogonal}$$

⑤ Eigen Vector of Symmetric Matrix Corresponding to different eigen Values are Orthogonal

Example: find the eigen Values Vectors and the Corresponding ~~of the~~ normalised eigen Vector Vector of the matrix given as

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Soln

$$A = 1\begin{vmatrix} -2 & -1 \\ -1 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix}$$

$$A = 1\Big((1)(-2) - (-1)(-1)\Big) - 3\Big(3(1) - (0)(-1)\Big) + 0\Big(3(-1) - (0)(-2)\Big)$$

$$A = 1(-2 - 1) - 3(3 - 0) + 0(-3 - 0) = 0$$

$$A = 1(-3) - 3(3) + 0(-3) = 0$$

$$A = -3 - 9 + 0$$

$$A = 12$$

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

but $|A - \lambda I_3| = 0$

$$\left|\begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right| = 0$$

$$\begin{pmatrix} 1-\lambda & 3 & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} = 0 \Rightarrow 1-\lambda\left[(-2-\lambda)(1-\lambda)-1\right] - 3\left[(3)(1-\lambda)-0\right] + 0 = 0$$

$$1-\lambda\left[-2+2\lambda-\lambda+\lambda^2-1\right] - 3\left[3-3\lambda\right] = 0$$

$$1-\lambda\left[-3+\lambda+\lambda^2\right] - 3\left[3-3\lambda\right] = 0 \Rightarrow$$

$$-3+\lambda+\lambda^2+3\lambda-\lambda^2-\lambda^3-9+9\lambda = 0$$

like terms

~~$-\lambda^3+\lambda^2-\lambda^2+\lambda^2+\overset{3\lambda+}{9\lambda}-3-9=0$~~

~~$-\lambda^3+13\lambda$~~

$$-\lambda^3+\lambda^2-\lambda^2+\lambda+3\lambda+9\lambda-3-9=0$$

$$\Rightarrow -\lambda^3+13\lambda-12=0$$

multiply − by the both side

$$\lambda^3-13\lambda+12=0$$

Testing for $\lambda-1=0$ and show that $\lambda-1$ is a factor of $\lambda^3-13\lambda+12=0$

Now divide by $\lambda^3-13\lambda+12$ by $\lambda-1$

$$\begin{array}{r} \lambda^2+\lambda-12 \\ \lambda-1 \overline{\big)\ \lambda^3-13\lambda+12} \\ \underline{\lambda^3-\lambda^2} \\ \lambda^2-13\lambda \\ \underline{\lambda^2-\lambda} \\ -12\lambda+12 \\ \underline{-12\lambda+12} \\ 0 \end{array}$$

∴ factoring

factoring $\lambda^2+\lambda-12=0$

$\lambda^2-3\lambda+4\lambda-12=0$

$\lambda(\lambda-3)+4(\lambda-3)=0$

$(\lambda+4)(\lambda-3)$ ——— ① $\Rightarrow \lambda_2=-4$, and $\lambda_3=3$

but the factor of $\lambda^3-13\lambda+12$ is $\lambda-1$ —— ② ∴ $\lambda_1=1$

$\lambda = 1, -4, 3$

for $\lambda=1$ $x_1$

$Ax=\lambda x$

$(A-\lambda I)x=0$

1- 1

$$\begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Note that we have gotten already

$$\begin{pmatrix} 1-\lambda & 3 & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} = 0$$

$$\left.\begin{aligned} x_1 + 3x_2 + 0 &= 0 \\ 3x_1 - 2x_2 - x_3 &= 0 \\ 0 - x_2 + x_3 &= 0 \end{aligned}\right\} \text{—} ①$$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

when $\lambda_1 = 1$, the above becomes

$$(A - \lambda I)x = 0$$

$$\begin{pmatrix} 1-1 & 3 & 0 \\ 3 & -2-1 & -1 \\ 0 & -1 & 1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 3 & 0 \\ 3 & -3 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

The eigen Vector equations gives the linear eqtns.

$3x_2 = 0$ —— (*)

$3x_1 - 3x_2 - x_3 = 0$ —— (**)

$-x_2 + \textcircled{x_2} = 0$ —— (***)

multiply - by [illegible]

$x_2 = 0$

$3x_1 - x_3 = 0$

$x_3 = 3$ (for non-trivial soln)

$x_1 = 1$

$$x = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

This is the eigen Vector corresponding $\lambda = 1$

If we Normalise x ~~x~~

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$= \sqrt{1^2 + 0^2 + 3^2}$$

$$= \sqrt{10} \quad \text{or} \quad 2\sqrt{5}$$

Con consequently the eigen Vector Corresponding to eigen Value ~~G~~ $e_1$ is

$$e_1 = \frac{1}{\|x\|} x$$

$$= \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

when $\lambda_2 = -4$

i.e

$$\begin{pmatrix} 1-\lambda & 3 & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix}$$

Now substitute ~~fo~~ $\lambda_2 = -4$

$$\begin{pmatrix} 1-(-4) & 3 & 0 \\ 3 & -2+4 & -1 \\ 0 & -1 & 1-(-4) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 5 & 3 & 0 \\ 3 & 2 & -1 \\ 0 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$5x_1 + 3x_2 = 0$

$3x_1 + 2x_2 - x_3 = 0$

$-x_2 + 5x_3 = 0$

for non-trivial soln

$x_2 = 5, x_3 = 1$

$x_1 = -3$

$$\therefore x = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

This is the eigen Value corresponding to $-4$

The Norm of the above is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

[illegible]

$$= \sqrt{(-3)^2 + (1)^2 + (5)^2}$$

~~$= \sqrt{10}$~~

$$\|x\| = \sqrt{(-3)^2 + (5)^2 + (1)^2}$$

$$= \sqrt{35}$$

The Corresponding Normalised eigen Vectors

$$e_1 = \frac{1}{\|x\|} \, x$$

$$= \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$e_2 = \frac{1}{\sqrt{35}} \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

$e_2 =$

find when $\lambda_3 = 3$ and $e_3$ at prep

∴ Since $\lambda_3 = 3$

$$\begin{pmatrix} 1-\lambda & 3 & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} = 0 \longrightarrow \begin{pmatrix} 1-3 & 3 & 0 \\ 3 & -2-3 & -1 \\ 0 & -1 & 1-3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -2 & 3 & 0 \\ 3 & -5 & -1 \\ 0 & -1 & +2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$-2x_1 + 3x_2 = 0$ — ①

$3x_1 - 5x_2 - x_3 = 0$ — ②

$-x_2 - 2x_3 = 0$ — ③

from ③ $x_3 = 1$ — ④ for non-trivial soln

$-x_2 - 2(1) = 0$

$-x_2 - 2 = 0$

∴ $x_2 = -2$ — ⑤

putting ④ and ⑤ into ②

$3x_1 - 5(-2) - (1) = 0$

$3x_1 + 10 - 1 = 0$

$3x_1 + 9 = 0$

$\frac{3x_1}{3} = \frac{-9}{3} \Rightarrow x_1 = -3$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$$

$$\text{ie } x = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$$

– this is the ~~eigen~~ Value Corresponding to $\lambda_3 = 3$ Vector

Taking the norm of the eigen Vector

$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{(-3)^2 + (-2)^2 + (1)}^{1} = \sqrt{14}$ ///

Consequently, the ~~Corresponding~~ eigen Vector corresponding Comes to eigen Value

$e_3 = \frac{1}{\|x\|} x \Rightarrow e_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$, we can write that

$$\Rightarrow e_3 = \begin{pmatrix} \frac{-3}{\sqrt{14}} \\ \frac{-2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix}$$

It may be noted that

~~$\langle e_1|$~~ $\langle e_1 | e_2 \rangle = 0 = \langle e_1 | e_3 \rangle = \langle e_2 | e_3 \rangle$

Consequently, the subset of the eigen Vector of A given by $\{e_1, e_2, e_3\}$ is orthonormalised. It is also trivially Complete. Consequently, every Vector $x \in R^3$ has a fourier Series expansion w.r.t the subset. also this subset is a basis for $R^3$.

If $w \in R^3$ and is given by $w = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, then the fourier Coefficients are given as

$\langle w | e_1 \rangle$ & $\langle w | e_2 \rangle$ & $\langle w | e_3 \rangle$

solving for $\langle w | e_1 \rangle$ ——— (*)

$\langle w | e_i \rangle = w_1 e_1 + w_2 e_2 + w_3 e_3$

Since ~~expanding~~ $e_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{pmatrix}$

as

on substituting into (*) above

$$\langle w | e_1 \rangle = \left\langle \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \middle| \begin{pmatrix} \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{pmatrix} \right\rangle = 2\left(\frac{1}{\sqrt{10}}\right) + 3(0) + (-4)\left(\frac{3}{\sqrt{10}}\right)$$

$$= \frac{2}{\sqrt{10}} - \frac{12}{\sqrt{10}} = \frac{-10}{\sqrt{10}}$$

solving ~~$e_2$~~ $\langle w|e_2 \rangle$ —— ①

where $w = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

and $e_2 = \frac{1}{\sqrt{35}} \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-3}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \\ \frac{1}{\sqrt{35}} \end{pmatrix}$

$e_2 =$

~~on~~ on substituting into eqn ①

$$\langle w|e_2 \rangle = \left\langle \begin{matrix} 2 \\ 3 \\ -4 \end{matrix} \middle| \begin{matrix} \frac{-3}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \\ \frac{1}{\sqrt{35}} \end{matrix} \right\rangle = 2\left(\frac{-3}{\sqrt{35}}\right) + 3\left(\frac{5}{\sqrt{35}}\right) - 4\left(\frac{1}{\sqrt{35}}\right)$$

$$= \frac{-6}{\sqrt{35}} + \frac{15}{\sqrt{35}} - \frac{4}{\sqrt{35}}$$

$$\frac{-6+15+4}{\sqrt{35}} = \frac{5}{\sqrt{35}}$$

Similarly solving for $\langle e_3\, w|e_3 ]$

we were given $w = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ and we have seen that $e_3 = \begin{pmatrix} \frac{-3}{\sqrt{14}} \\ \frac{-2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix}$

$$\therefore \langle w|e_3 \rangle = \left\langle \begin{matrix} 2 \\ 3 \\ -4 \end{matrix} \middle| \begin{matrix} -3 \\ -2 \\ 1 \end{matrix} \right\rangle = -3$$

$$\langle w|e_3 \rangle = \left\langle \begin{matrix} 2 \\ 3 \\ -4 \end{matrix} \middle| \begin{matrix} \frac{-3}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{matrix} \right\rangle = 2\left(\frac{-3}{\sqrt{14}}\right) + 3\left(\frac{-2}{\sqrt{14}}\right) - 4\left(\frac{1}{\sqrt{14}}\right)$$

$$= \frac{-6}{\sqrt{14}} - \frac{6}{\sqrt{14}} \; \frac{-4}{\sqrt{14}}$$

LCM

$$= \frac{-16}{\sqrt{14}}$$

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* Consequently the fourier series of ~~the basis~~ w (The summation of all the values)

$$\langle w|e_i\rangle = -\frac{10}{\sqrt{10}}e_1 + \frac{5}{\sqrt{35}}e_2 + \frac{16}{\sqrt{14}}e_3$$

Exercise: find the eigen Value and the corresponding normalizing eigen Vectors for the following operators

① $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

② $B - \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$

3 $A - \begin{pmatrix} 5 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

## Cayley – Hamilton Theorem

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Consequently the fourier series of ~~the basis~~ w

* (The Combination of all the Values)

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## Cayley – Hamilton Theorem

solⁿ to Exercise (prep)
find the eigenvalue λ

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

using the formula
$|A - \lambda I_3| = 0$

$$\left|\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right| = 0$$

$$\begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = 0$$

$2-\lambda\left[(2-\lambda)(1-\lambda)\right] - 1\left[(1)(1-\lambda)\right] + 1\left[0\right] = 0$

$2-\lambda\left[2 - 2\lambda - \lambda + \lambda^2\right] - 1\left[1-\lambda\right]$

$2-\lambda\left[2 - 3\lambda + \lambda^2\right] - 1 + \lambda] = 0$

$4 - 6\lambda + 2\lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda = 0$

like terms

$-\lambda^3 + 2\lambda^2 + 3\lambda^2 - 6\lambda - 2\lambda + \lambda + 4 - 1 = 0$

$-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$

multiply − all through, we have

$* \quad \lambda^3 \cancel{+7\lambda^2 +} 7\lambda - 3 = 0$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

factorise
$\lambda(\lambda - 1)$

Evaluating the eqtn ~~Schoo~~ above with almighty formula

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

$$\frac{-5 \pm \sqrt{b^2 - 4ac}}{2a}$$

from the above

$a = 1, b = -5, c = 3$

$\therefore -(-5) \pm \sqrt{(-5}$

long division
Testing for factor, that ~~[illegible]~~ makes the eqtn 0
$\lambda - 1 = 0 \Rightarrow \lambda = 0$
ie
$(1)^3 - 5(1)^2 + 7(1) - 3 = 0$
$\therefore \lambda - 1 = 0$ is a factor

$$\begin{array}{r l}
 & \lambda^2 - 4\lambda + 3 \\
\lambda - 1 & \overline{)\lambda^3 - 5\lambda^2 + 7\lambda - 3} \\
 & \lambda^3 - \lambda^2 \\
 & -4\lambda^2 + 7\lambda - 3 \\
 & -4\lambda^2 + 4\lambda \\
 & 3\lambda - 3 \\
 & 3\lambda - 3 \\
 & 0 \quad 0
\end{array}$$

we have $\lambda^2 - 4\lambda + 3$

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Resolving with almighty formula

$\lambda^2 - 4\lambda + 3$

the

$$-\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here, $a = 1, b = -4, c = 3$

$$\therefore \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$$

$$4 \pm \sqrt{\frac{16 - 12}{2}} = 4$$

$$4 \pm \sqrt{\frac{4}{2}} = \frac{4 \pm 2}{2}$$

$$\therefore \frac{4+2}{2} \text{ or } \frac{4-2}{2}$$

$$\frac{6}{2} \text{ or } \frac{2}{2}, \therefore 3 \text{ or } 1$$

we conclude that

$\lambda_1 = 1$ $\lambda_2 = 1$ and $\lambda_3 = 3$

Note $\lambda_1$ was found by the factor

02/10/24

## CAYLEY-HAMILTON THEOREM

It states that every square matrix satisfies its charateristic equation

~~# this state every~~ :

let assume a square matrix

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$ satisfy

$$\lambda^3 - 2\lambda^2 + \lambda - 4 = 0$$

Mathematical statement of Cayley - Hamilton theorem.

let

$$|A - \lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} \dots + a_n)$$

be xtics polynomial of ~~by~~ $n \times n$ matrix

$A = a_{ij}$, then the matrix equation

$$X^n + a_1X^{n-1} + a_2X^{n-2} + \dots + a_nI = 0,$$

is Satisfied by $X = A$ ie

$$A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI = 0 \rightarrow (i)$$

Proof : (i)

It is well-known, that the elements of the matrix [A - λI] are at most of the first degree in λ, it follows that the elements of ~~adjoint~~ of Adj.$(A - \lambda I)$ are at most of degree of $n - 1$ in λ. Thus Adj$(A - \lambda I)$ may be ~~word~~ written as the adjoint matrix polynomials in λ given as.

$$\text{adj}\,(A - \lambda I) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \cdots + B_{n-1}$$ ——— ②

where $B_0, B_1 \cdots B_{n-1}$ are $n \times n$ matrixes with elements been polynomials in $\lambda$. ~~we k~~

we know that

$$(A - \lambda I)\,\text{adj}\,(A - \lambda I) = |A - \lambda I|\, I$$ ——— ③

comparing ③ with above equations ②

④

$$(A - \lambda I)(B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \cdots - B_{n-1}) = (-1)^n (\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n) I$$

equating coefficient of like powers of $\lambda$ on both sides of eqtn ④

$$-B_0 = (-1)^n I$$

$$AB_0 = IB_1 = (-1)^n a_1 I$$

$$AB_1 - IB_2 = (-1)^n$$

$$AB_{n-1} = (-1)^n a_n I$$ ——— ⑤

multiplying eqn ⑤ with by $A^n, A^{-1}, \cdots, I$ respectively and adding, we have:

$$0 = (-1)^n [A^n$$

$\cancel{x}A x^{-1} SA$

which gives $\qquad \frac{3\cancel{A}}{\cancel{A}} = 3$

$$A^n + a_1 A^{n-1} + \dots + a_n I = 0 \quad \text{——} ⑥$$

equation ⑥ is the Cayley Hamilton equation.

**Example:**

Suppose A is an $m \times m$ matrix with the characteristic equation given as:

$$\lambda^3 - 2\lambda^2 + 3\lambda - 4 = 0 \quad \text{——} ①$$

Then according to Cayley Hamilton's theorem, eqtn ① is satisfied by A such that

$$A^3 - 2A^2 + 3A - 4I = 0 \quad \text{——} ②$$

To determine $A^{-1}$, multiply eqtn ② by $A^{-1}$

$$A^2 - 2A + 3I - 4A^{-1} = 0$$

$$A^2 - 2A + 3I = 4A^{-1}$$

$$\therefore \cancel{A}\ A^{-1} \cancel{= A^2} = \frac{1}{4}\left(A^2 - 2A + 3I\right)$$

**Example ②:**

Verify Cayley Hamilton's theorem for the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

soln

$$|A - \lambda I| = 0 \quad \text{——} ⊛$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

~~A = A λ~~

∴ And from eqtn (*), we have; $\left|\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right| = 0$

$$\left|\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right| = \begin{pmatrix}1-\lambda & 2\\ 2 & -1-\lambda\end{pmatrix} \Rightarrow 0$$

$$[(1-\lambda)(-1-\lambda)] - [(2)(2)] \Rightarrow -1-\lambda+\lambda+\lambda^2 - 4 = 0$$

$$-1+\lambda^2-4=0 \quad -5+\lambda^2=0 \Rightarrow \lambda^2$$

$$\lambda^2 - 5 = 0$$

by the Cayley Hamilton theorem, we have

$$A^2 - 5I = 0$$

$$A^2 = A \cdot A = \begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix} = \cancel{1+4}$$

$A^2 =$ which gives $\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}$

(*) the above result was gotten by solving

$$A^2 = A \cdot A = \begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$$

use d first row to multiply the first column. And use the first row to multiply the second column.

Then, use the second row to multiply the first column. And use the second row to multiply the second column

$$= \begin{matrix}1\times1 + 2\times2 & 1\times2 + (2)(-1)\\ 2\times1 + (-1)\times(2) & (2)(2) + (-1)(-1)\end{matrix}$$

$$\therefore = \left.\begin{matrix}1+4 & 2-2\\ 2+2 & 4+1\end{matrix}\right\} = \begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}$$

$$\therefore A^2 = A \cdot A = \begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}$$

p.t.o

Rodrigues

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} + \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \odot$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \text{ ——— } ②$$

from eqn ① and ② Cayley Hamilton theorem is satisfied.

$A^2 - 5I = 0$

multiply it by $A^{-1}$

$A - 5A^{-1} = 0$

$A^{-1} = \frac{1}{5}A$

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix}$$

## The Space $C[0, \pi]$

Here, we shall study the basis generated by the linear operator

$A = \frac{d^2}{dx^2}$ for a vector of all piecewise continuous and ~~smot~~ smooth complex valued function defined on the interval $[a, b]$

Definition: Suppose $C[0, \pi]$ is the set of all piecewise, continuous and smooth complex valued functions defined on the interval $[0, \pi]$.

Let $+$ be the operation of addition defined on

~~$c[0,\pi]$ such that~~ $c[0,\pi]$; $\forall y,z \in c[0,\pi]$

$(y+z)(x) = y(x) + z(x)$; $\forall x \in [0,\pi]$.

~~let~~

let $\cdot$ be the operation of multiplication defined on ~~$c[0,\pi]$~~ $c[0,\pi]$, $\forall y \in c[0,\pi]$ and $\lambda \in C$ such that

$(\lambda y)(x) = \lambda y(x)$; $\forall x \in [0,\pi]$

Then, the operation $+, \cdot$ makes $c[0,\pi]$ a vector space over a complex field $C$

## Wronskian Of a Subset

let $\{y,z\}$, $c[0,\pi]$ then the determinant

$$\begin{vmatrix} y(x) & z(x) \\ y'(x) & z'(x) \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} y(x) & z(x) \\ \frac{dy}{dx} & \frac{dz}{dx} \end{vmatrix}$$

where ~~prime~~ prime, denotes a first derivative is called the ~~wr~~ Wronskian of a the subset and denoted by

$W\{y,z\}$

Similarly, if $\{y,z,w\}$ is a subset of $c[0,\pi]$, then the determinant ~~the~~

$$\begin{vmatrix} y(x) & z(x) & w(x) \\ y'(x) & z'(x) & w'(x) \\ y''(x) & z''(x) & w''(x) \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} y(x) & z(x) & w(x) \\ \frac{dy}{dx} & \frac{dz}{dx} & \frac{dw}{dx} \\ \frac{d^2y}{dx^2} & \frac{d^2z}{dx^2} & \frac{d^2w}{dx^2} \end{vmatrix}$$

is called the wronskian of the subset, where each prime denotes differentiation and this is denoted by

$w\{y,z,w\}$

(i) Example:

Determine the Wronskian of the subset of $\{y,z\}$ of $c[0,\pi]$ defined by:

$y(x) = \cos x$, $z(x) = \sin x$

<u>Soln</u>

$$W\{y, z\} = \begin{vmatrix} y(x) & z(x) \\ y'(x) & z'(x) \end{vmatrix}$$

$$W\{\cos x, \sin x\} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1$$ (it is linearly independent since $W\{y, z\} \neq 0$)

fr **Prep**

① find the Wronskian of the subset ~~given~~ by $\{y, z, w\}, c[0, \pi]$ given by $y(x) = 1$, $z(x) = x$, $w(x) = x^2$

Ans (2)

② find the wronskian of the subset of $\{y, z, w\}, c[0, \pi]$ given by $y(x) = 1 - x$, $z(x) = x - x^2$, $w(x) = 1 - x^2$

Ans: (0)

**Solution to prep ①**

① $$W\{y, z, w\} = \begin{vmatrix} y(x) & z(x) & w(x) \\ y'(x) & z'(x) & w'(x) \\ y'(x) & z''(x) & w''(x) \end{vmatrix} = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} \end{array}$$

$$= 1\left[(1)(2) - (0)(2x)\right] - x\left[(0)(2) - (0)(2x)\right] + x^2\left[(0)(0) - (0)(1)\right]$$

$$= 1[2] - x[0] + x^2[0] = 2$$

$\therefore W\{y, z, w\} = 2$ (it is linearly independent)

② $$W\{y, z, w\} = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} y(x) & z(x) & w(x) \\ y'(x) & z'(x) & w'(x) \\ y''(x) & z''(x) & w''(x) \end{vmatrix} \end{array} = \begin{vmatrix} 1-x & x-x^2 & 1-x^2 \\ -1 & 1-2x & -2x \\ 0 & -2 & -2 \end{vmatrix}$$

$$= 1-x\left[(1-2x)(-2) - (-2)(-2x)\right] - x-x^2\left[(-1)(-2) - (0)(-2x)\right] + 1-x^2\left[(-1)(-2) - (0)(1-2x)\right]$$

$$= 1-x\left[-2 + \cancel{4x} - 4x\right] - x - x^2\left[1 - 0\right] + 1 - x^2\left[1 - 0\right]$$

~~$W[y,z,w] = [-x[-2] - x - x^2[1] + 1-x^2[1]$~~

~~$= [-2-2x - x - x^2 + 1 - x^2] = [-1 - 3x - 2x^2$~~

Remark: It is obvious how to ~~explain~~ extend the definition of the wronskian of any subset of $C[0,\pi]$

Theorem:

The quantity $A$ defined by

$$Ay = \frac{d^2}{dx^2} y(x), \quad \forall y \in C[0,\pi], x \in [0,\pi] \text{ is}$$

a linear operator on $C[0,\pi]$ $\qquad -1-2$

## Soln to prep ②

$$W\{y,z,w\} = \begin{vmatrix} y(x) & z(x) & w(x) \\ y'(x) & z'(x) & w'(x) \\ y''(x) & z''(x) & w''(x) \end{vmatrix} = \begin{vmatrix} \overset{+}{1-x} & \overset{-}{x-x^2} & \overset{+}{1-x^2} \\ -1 & 1-2x & -2x \\ 0 & -2 & -2 \end{vmatrix}$$

$$= 1-x \begin{vmatrix} 1-2x & -2x \\ -2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} -1 & -2x \\ 0 & -2 \end{vmatrix} + 1-x^2 \begin{vmatrix} -1 & 1-2x \\ 0 & -2 \end{vmatrix}$$

$$= (1-x)\left[(1-2x)(-2) - (-2)(-2x)\right] - (x+x^2)\left[(-1)(-2) - 0\right] + (1-x^2)\left[(-1)(-2) - 0\right]$$

$$= \{(1-x)(-2+4x-4x)\} - (x+x^2)(2) + (1-x^2)(2)$$

$$= 1-x(-2) + (x^2+x)(2) + 1-x^2(2)$$

$$= -\cancel{2} + \cancel{2x} + \cancel{2x^2} - \cancel{2x} + \cancel{2} - \cancel{2x^2} = \boxed{0}$$

$\therefore W\{y,z,w\} = 0$ (we conclude that it is linearly dependent.)

We define linear operator $A$

Proof:

If $y \in C[0,\pi]$, $Ay \in C[0,\pi]$

If $z \in C[0,\pi]$ also, then $A(y+z) = \frac{d^2}{dx^2}[y(x) + z(x)]$

$$= \frac{d^2}{dx^2} y(x) + \frac{d^2}{dx^2} z(x)$$

$\therefore$ Sat $Ay + Az$

Similarly, if $\lambda \in C[0,\pi]$

$$A(\lambda y) = \frac{d^2}{dx^2}(\lambda y(x))$$

$$= \lambda \frac{d^2}{dx^2} y(x)$$

$$= \lambda Ay$$ $\therefore$ A is a linear Operator

## EIGEN VALUES & EIGEN VECTORS

Consider the eigen vc problem for the linear Operator A defined as

$Ay = -\lambda y$ Or

$y''(x) = -\lambda y(x)$ —— ①

Subject to the Conditions of

$y'(0) = 0$ —— ② and

$y'(\pi) = 0$ —— ③

The linearly independent soln of Eqtn ①

$$y(x) = \begin{cases} \cos(\sqrt{\lambda} x) \\ \sin(\sqrt{\lambda} x) \end{cases}$$

The sine function Cannot satisfy the two Conditions

$y(x) = \sin(\sqrt{\lambda} x)$

$y'(x) = \sqrt{\lambda} \cos(\sqrt{\lambda} x)$

at $x = 0$

$y'(0) = \sqrt{\lambda} \cos(0) = \sqrt{\lambda}$ //

but the cosine functions satisfies $y'(0) = 0$, Identically it follows from equation ③ that

$\sin(\sqrt{\lambda}\pi) = 0$

i.e

$\sqrt{\lambda}\pi = n\pi$ ; $n = 0, 1, 2, 3 \ldots$

$\Rightarrow \lambda = n^2$ ; $n = 0, 1 \ldots$

$y(x) = \cos(\sqrt{\lambda} x)$

$y'(x) = -(\sqrt{\lambda} \sin(\sqrt{\lambda} x)$

$= \frac{-\sqrt{\lambda} \sin(\sqrt{\lambda}\pi)}{-\sqrt{\lambda}} = \frac{0}{-\sqrt{\lambda}}$

$= \sin(\sqrt{\lambda}\pi) = 0$

$\sqrt{\lambda}\pi = \sin^{-1}(0)$

$\sqrt{\lambda}\pi = n\pi$

$\therefore$ ~~$\lambda = n^2$~~ Squaring both sides yields

$\lambda = n^2$

which are the eigen values, the corresponding eigen values or eigen functions denoted by $\{y_n\}$ ~~[illegible]~~ sequence $\{y_n\}_{n=0}^{\infty}$

$y = n$

$(\Pi) =$

## INNER PRODUCT

The product for the space $C[0, \pi]$ is given by

$$\boxed{\langle y_1 | y_2 \rangle = \int_0^{\pi} \bar{y}_1(x)\, y_2(x)\, dx}$$

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It maybe noted that, this inner product induces a corresponding norm given by the

$$\|y\|^2 = \int_0^{\pi} y^2(x)dx$$

under which the space $C[0,\pi]$ is a Hilbert space.

## FOURIER COSINE SERIES

Since the space $C[0,\pi]$ is a Hilbert space, it follows that, the subset $\{e_n\}_{n=0}^{\infty}$ is defined by

$$e_n(x) = \begin{cases} \pi^{-\frac{1}{2}} ; n=0 \\ (\frac{\pi}{2})^{-\frac{1}{2}} \cos(nx) ; n>0 \end{cases}$$ is an Orthonormal basis.

Consequently, every Element $y \in C[0,\pi]$ has a fourier series expansion ~~$y(x) = \sum_{n=0}^{\infty} \langle$~~

$$\boxed{y(x) = \sum_{n=0}^{\infty} \langle y|e_n \rangle e_n(x)}$$ — ① This is series is

called the fourier cosine series for $y$. where $\boxed{\langle y|e_n \rangle = \int_0^{\pi} \bar{y}(x) e_n dx}$

Example

find the fourier ser Cosine Series expansion of the function

① $y(x) = x^2$ wrt

$$e_n = \begin{cases} \pi^{-\frac{1}{2}} ; n=0 \\ (\frac{\pi}{2})^{-\frac{1}{2}} \cos(nx) ; n>0 \end{cases}$$

Soln

$y(x) = x^2 \Rightarrow \bar{y}(x) = x^2$

at $n=0$

$e_0 = (\pi)^{-\frac{1}{2}}$

$\therefore$ ~~$\langle y|e_0$~~ $\rangle$, using the inner product formula

$$\langle y|e_n \rangle = \int_0^{\pi} \bar{y}(x) e_n dx$$

|  | D | I |
|---|---|---|
| + | $x^2$ | $\cos x$ |
| − | $2x$ | $\sin x$ |
| + | $2$ | $-\cos x$ |
| − | $0$ | $-\sin x$ |
| + | $0$ | $\cos x$ |

$\therefore \langle y|e_0\rangle = \int_0^{\pi} \bar{y}(x)\, e_0\, dx$

On substitution

$= \int_0^{\pi} x^2 (\pi)^{-1/2} dx = (\pi)^{-1/2} \int_0^{\pi} x^2 dx$

$= (\pi)^{-1/2} \left[\frac{x^3}{3}\right]_0^{\pi}$

$= (\pi)^{-1/2} \left[\pi^3/3\right] - \left[\cancel{\frac{0^3}{3}}\right] \to 0$

$= \frac{1}{\sqrt{\pi}} \times \frac{\pi^3}{3} = \frac{\pi^3}{3\sqrt{\pi}}$ or $\frac{\pi^2\sqrt{\pi}}{3}$

$\langle y|e_0\rangle = \frac{\pi^2\sqrt{\pi}}{3} \Rightarrow \langle y|e_0\rangle e_n(x) = (\pi)^{-1/2} \times \frac{\pi^2\sqrt{\pi}}{3} \Rightarrow \frac{1}{\sqrt{\pi}} \times \frac{\pi^2\sqrt{\pi}}{3}$

$\therefore y(x) = \frac{\pi^2}{3}$ //

**Prep** Determine $e_1$, $e_2$, and and $e_4$ for the above question

Solution

from the given question $e_n = \begin{cases} \pi^{-1/2}; & n=0 \\ (\pi/2)\cos(nx); & n>0 \end{cases}$

using the formular

$\langle y|e_n\rangle = \int_0^{\pi} \bar{y}(x)\, e_n\, dx$ ——— ①

Here ~~x~~ $\bar{y}(x) = x^2$

~~$e_1$ $y(x)$~~ $e_n = (\pi/2)\cos(nx) = \cancel{x}$

at $n = 1$, we have from the above ~~from~~ that

$\langle y|e_1\rangle = \int_0^{\pi} x^2\, \pi/2\, \cos x\, dx$ since n=1

$\langle y|e_1\rangle = \frac{\pi}{2}\int_0^{\pi} x^2 \cos x\, dx = \frac{\pi}{2}$

$= \frac{\pi}{2}\left[x^2\sin x + 2x\cos x - 2\sin x + \cancel{\cos x}\right]_0^{\pi}$

$= \frac{\pi}{2}\left[\left(\pi^2\cancel{\sin\pi} + 2\pi\cos\pi - \cancel{2\sin\pi} - \cancel{\cos\pi}\right) - \left(0^2\cancel{\sin(0)} + 2(0)\cancel{\cos(0)} - 2\cancel{\sin(0)} - \cancel{\cos 0}\right)\right]$

$= \frac{\pi}{2}\left[-2\pi\right]$ ~~$\cos\pi$~~ $] = \frac{\pi}{2}\left[-2\pi \cdot \cancel{\cos\pi}\right] =$

$$y(x) = \sum_{n=0}^{\infty} \langle y|e_n \rangle e_n = \langle y|e_0 \rangle e_0 + \langle y|e_1 \rangle e_1 + \langle y|e_2 \rangle e_2 + \cdots$$

is $y(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2^2}{n^2} (-1)^n \cos(nx)$

It may be noted that

$$\|y\|^2 = \int_0^{\pi} x^2 \, dx = \frac{\pi^3}{3}$$

## FOURIER SINE SERIES EXPANDING

The eigen problem for the linear operator $A = \frac{d^2}{dx^2}$ on the space $C[0,\pi]$ given by

$$y''(x) = -\lambda y(x) \; ; \; 0 \le x \le \pi$$

Subject to the Conditions

$$y(0) = 0$$
$$y^{*}(\pi) = 0$$

generates an orthonormal basis for $C[0,\pi]$ given by $\{e_n\}_{n=1}^{\infty}$

where $e_n(x) = \left(\frac{2}{\pi}\right)^{1/2} \sin(nx)$. The fourier series expansion wrt the orthonormal basis is called fourier sine series.

Example.

find the fourier sine series expansion of the function with respect to the basis

$$y(x) = x(\pi - x) \; ; \; 0 \le x \le \pi$$

$$e_n(x) = \left(\frac{2}{\pi}\right)^{1/2} \sin(nx)$$

Soln

The fourier coefficient wrt the basis are given as

$$\langle y|e_n \rangle = \int_0^{\pi} y(x) \, e_n(x) \, dx$$

on substitution

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[\pi\int_0 x\sin x\,dx - \int x^2\sin x\,dx\right]$$

$$\langle y|e_n\rangle = \int_0^{\pi} x(\pi - x)\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin(nx)\,dx$$

| D | I |
|---|---|
| $+\ x$ | $\sin x$ |
| $-\ 1$ | $-\cos x$ |
| $+\ 0$ | $-\sin x$ |

for $n=1$

$$\langle y|e_1\rangle = \int_0^{\pi} x(\pi - x)\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin x\,dx$$

| D | I |
|---|---|
| $+\ x^2$ | $\sin x$ |
| $-2x$ | $-\cos x$ |
| $+\ 2$ | $-\sin x$ |
| $-\ 0$ | $\cos x$ |

After solving, we have $\left(\frac{2}{\pi}\right)^{\frac{1}{2}} 2^2$

**Solution**

$$\langle y|e_1\rangle = \int_0^{\pi} x(\pi - x)\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin x\,dx$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\pi} (x\pi - x^2)(\sin x\,dx) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} ~~\pi \int_0^{\pi} (x\pi - x^2)(\sin x\,dx)~~$$

$$\langle y|e_1\rangle = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\pi\int_0^{\pi} x\sin x\,dx - \int_0 x^2\sin x\,dx$$

we can now integrate.

$$\langle y|e_1\rangle = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\pi\Big[(-x\cos x + \sin x)\Big]_0^{\pi} - \Big[-x^2\cos x + 2x\sin x + 2\cos x\Big]_0^{\pi}$$

$$\langle y|e_1\rangle = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Big\{\pi\Big[(-\pi\cos\pi + \cancel{\sin\pi}^{0}) - (-0\cancel{\cos(0)}^{0} + \cancel{\sin(0)}^{0})\Big] - \Big[(-\pi^2\cos\pi + \cancel{2\pi\sin\pi}^{0} + 2\cos\pi) - (-0^2\cancel{\cos(0)}^{0} + 2(0)\cancel{\sin(0)}^{0} + 2\cos(0))\Big]\Big\}$$

~~that is~~ because $\cos\pi = -1$ and $\sin\pi = 0$

$$\langle y|e_1\rangle = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Big\{\pi\big[-\pi(-1)\big] ~~- [(-2)] - (2)~~\Big] - \big[-\pi^2 - 2 - 2\big]$$

$$\langle y|e_1\rangle = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Big\{\pi[\pi] - [\pi^2 - 2 - 2\Big\} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Big\{\pi^2 + \pi^2 + 2 + 2\Big\}$$

$$\langle y|e_1\rangle = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\cancel{\pi^2}^{0} - \cancel{\pi^2}^{0} + 2 + 2\right)$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} 4 \quad \text{or} \quad \langle y|e_1\rangle = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} 2^2$$

Similarly: at $n=2$

$$\langle y|e_2\rangle = \int_0^{\pi} x(\pi - x)\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin 2x\,dx$$

After solving, we have: $\langle y|e_2\rangle = 0$

and Determining $e_3$, $e_4$ and $e_5$

for the first three Values, we have.

fourier series $\sum_{n=1}^{k} \langle y | e_n \rangle = \langle y | e_1 \rangle + \langle y | e_2 \rangle + \dots$

$$= \frac{8}{\pi} \left[ \sin x + \frac{\sin(3x)}{3^3} + \frac{\sin(5x)}{5^3} + \dots \right]$$

at $n = 2$

| D | I |
|---|---|
| $+x$ | $\sin 2x$ |
| $-1$ | $-\frac{1}{2}\cos 2x$ |
| $+0$ | $-\frac{1}{4}\sin 2x$ |

| D | I |
|---|---|
| $+x^2$ | $\sin 2x$ |
| $-2x$ | $-\frac{1}{2}\cos 2x$ |
| $+2$ | $-\frac{1}{4}\sin 2x$ |
| $-0$ | $\frac{1}{8}\cos 2x$ |

The corresponding norms of is

$$\|y\|^2 = \int_0^{\pi} [x(\pi - x)]^2 dx$$

$$\|y\|^2 = \frac{\pi^5}{30}$$

$$\|y\| = \sqrt{\pi^5/30}$$

---

At $e_2$ ie at $n=2$

# LEGENDRE FUNCTIONS

The Legendre differential equation is given as:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{Or}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0 \quad \text{Or}$$

$$(1-x^2)y''(x) - 2xy'(x) + n(n+1)y = 0$$

The general solution of the Longer Legendre's equation is given as:

$$y = AP_n(x) + BQ_n(x)$$

where $P_n(x)$ and $Q_n(x)$ are two independent solutions of the Legendre's equation. The Legendre functions are at orthonormal basis for the space $C(-1,1)$ ~~and~~ of all piecewise continous and smooth real valued functions on the interval $(-1,1)$.

Theorem. The eigenfunctions of the Legendre differential equations corresponding to distinct eigen values are orthonormal with to wrt weight ~~function~~ 1

proofs:

Let $y_m$ ~~be eq~~ and $y_n$ be eigen function ~~be~~ of Legendre differential equations corresponding to distinct eigen value. Then by definition:

$$[(1-x^2)y_m'(x)]' + \lambda_m y_m(x) = 0 \text{ ——— } ①$$

and

$$\{(1-x^2)y_n'(x)]' + \lambda_n y_n(x) = 0 \text{ ——— } ②$$

multiply eqtn ① by $y_n(x)$ and ~~②~~ $y_m(x)$ and subtract gives

$$\bar{y}_m(x)\left[(1-x^2)y_n'(x)\right]' - y_m(x)\left[(1-x^2)y_m'(x)\right]' + (\lambda_n-\lambda_m)\bar{y}_m(x)\,y_n(x)=0 \quad ③$$

Integrating eqn ③ between −1 and 1

$$\int_{-1}^{1}\bar{y}_m(x)\left[(1-x^2)y_m'(x)\right]' - \int_{-1}^{1} y_m(x)\left[(1-x^2)y_m'(x)\right]' + \int_{-1}^{1}(\lambda_n-\lambda_m)\bar{y}_m(x)\,y_n(x)$$

$$y_m(x)\left[(1-x^2)y_n'(x)\right]^2\Big|_{-1}^{1} - \left[y_m(x)(1-x^2)y_m'(x)\right]_{-1}^{1} + \int_{-1}^{1}(\lambda_n-\lambda_m)\bar{y}_m(x)\,y_n(x)$$

$$\bar{y}_m y(x)\left[(1-1)y_m'(x)\right]^2 - \left[y_m(x)(1-1)y_m'(x)\right]^2 + \int_{-1}^{1}(\lambda_m-\lambda_n)\bar{y}_m(x)\,y_n(x)$$

$$y_m y(x)\left[(0)\bar{y}_m(x)\right]^2 - \left[y_m(x)\,(0)\,y_m'(x)\right]_{-1} \int_{-1}^{1}(\lambda_n-\lambda_m)\bar{y}_m(x)\,y_n(x)$$

$$(\lambda_n-\lambda_m)\int_{-1}^{1} y_m(x)\,y_n(x)\,dx = 0 \; ; \; \lambda_n \neq \lambda_m$$

$\therefore$

## ~~tat~~ INNER PRODUCT

As a result of orthogonality property or condition of the eigen functions of legendre's differential equation and the inner product on the space $C(-1,1)$ may be chosen as

$$\langle y|z\rangle = \int_{-1}^{1} \bar{y}(x)\, z(x)\, dx; \quad \forall y, z \in C(-1,1)$$

The inner product induces a corresponding norm is given by

$$\|y\|^2 = \int_{-1}^{1} y^2(x)\, dx; \quad \forall y \in C(-1,1)$$

Thus, making a space $C(-1,1)$ a Hilbert space.

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## LEGÆENDRE'S POLYNOMIALS $P_n(x)$

The legendre eigenproblem possess eigen value $\lambda$ given by,

$\lambda = n(n+1); \; n = 0, 1, 2 \ldots$

and the corresponding eigenfunction given as

$y(x) = P_n(x)$

$P_n(x)$ is the solution of the legendre's differential equation which is unity or 1 when $x = 1$.

It may be noted that, the first five legendre polynomials are given as

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3) \ldots$$

Generally

$$P_n(x) = \sum_{r=0}^{n} \frac{(-1)^r (2n-2r)!}{2^n . r! (n-r)! (r-2)!} x^{n-2r}$$

RODRIGUE'S FORMULA FOR GENERATING LEGENDRE'S POLYNOMIAL

The Rodriques for generating legendre polynomial is given as

$$\boxed{P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n} (x^2-1)^n}$$

when $n = 0$

$$P_0(x) = \frac{1}{2^0 . 0!} \frac{d^0}{dx^0} (x^2-1)^0 = \frac{1}{1} \cdot \frac{1}{1} = 1$$

i.e

$P_0(x) = 1$

when $n = 4$

$$P_4(x) = \frac{1}{2^4 . 4!} \frac{d^4}{dx^4} (x^2-1)^4$$

$$P_4(x) = \frac{1}{16 \cdot 24} \frac{d^4}{dx^4} \left[\frac{d}{dx} (x^2-1)^4\right]$$

$$= \frac{1}{16 \cdot 24} \frac{d}{dx^3}$$

$(x^2-1)^4$

$\{(x^2-1)(x^2-1)\}\{(x^2-1)(x^2-1)\}$

$(x^2-2x+1)(x^2-2x+1)$

$x^4-2x^3+x^2-2x^3+4x^2-2x$

$+x^2-2x+1$

$\Rightarrow$

$(x^2)^3$

$x^6$

Prep, determine $P_7(x)$, $P_6(x)$, $P_5(x)$

Example: show that

$$\int_{-1}^{1} P_n(x)\,dx = \left[\begin{array}{l} 0\,;\ n \neq 0 \\ \\ 2\,;\ n = 0 \end{array}\right.$$

from ou rodrigue's formula

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\int_{-1}^{1} P_n(x) = \frac{1}{2^n.n!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{1}{2^n.n!}\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right]_{-1}^{1}$$

$$= \frac{1}{2^n.n!}\left\{\left[\left(\frac{d^{n-1}}{dx^{n-1}}((1)^2-1)^{n-1}\right) - \left[\frac{d^{n-1}}{dx^{n-1}}((-1)^2-1)^{n-1}\right]\right]\right\}$$

$$= F\,\frac{1}{2^n.n!}\left(\cancel{\frac{d^{n-1}}{dx^{n-1}}(0)^{n-1}}\right)^{\to 0} - \frac{1}{2^n.n!}\left(\cancel{\frac{d^{n-1}}{dx^{n-1}}(0)^{n-1}}\right)^{\to 0}$$

$P_n(x) = 0$ both ie when $n \neq 0$

but when $n = 0$

$P_0(x) = 1$

$$\int_{-1}^{1} P_0(x)\,dx = \int_{-1}^{1} 1.\,dx = x\Big|_{-1}^{1} = (1)-(-1) = 2$$

So at $n = 0$ is 2 //

# Soln to prep

using Rodrigue's formula to solve for $P_4(x)$, $P_5(x)$, an $P_6(x)$ and $P_7(x)$

soln

from Rodrigue's formula, $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

at $n=4$

*

$$P_4(x) = \frac{1}{2^4 4!}\frac{d^4}{dx^4}(x^2-1)^4 \text{ ——— } ①$$

$$(x^2-1)^4 \Rightarrow [(x^2-1)(x^2-1)][(x^2-1)(x^2-1)]$$

$$= \{x^4 - x^2 - x^2 + 1\}\{x^4 - x^2 - x^2 + 1\} =$$

$=$ ~~$= x^8 - x \Rightarrow$~~ $[x^4 - 2x^2 + 1][x^4 - 2x^2 + 1]$

$$= x^8 - 2x^6 + x^4 - 2x^6 + 4x^4 - 2x^2 + x^4 - 2x^2 + 1$$

Rearranging, we have,

$$= x^8 - 2x^6 - 2x^6 + x^4 + 4x^4 + x^4 - 2x^2 - 2x^2 + 1$$

$$= x^8 - 4x^6 + 6x^4 - 4x^2 + 1 \text{ ——— } ②$$

putting ② into ①

$$P_4(x) = \frac{1}{2^4.4!}\frac{d^4}{dx^4}(x^8 - 4x^6 + 6x^4 - 4x^2 + 1)$$

$$P_4(x) = \frac{1}{16.24}\frac{d^3}{dx^3}(8x^7 - 24x^5 + 24x^3 - 8x)$$

$$P_4(x) = \frac{1}{384}\frac{d^2}{dx^2}(56x^6 - 120x^4 + 72x^2 - 8)$$

$$P_4(x) \frac{1}{384}\frac{d}{dx}(336x^5 - 420x^3 + 144x)$$

$$P_4(x) = \frac{1}{384}(1680x^4 - 1440x^2 + 144)$$

$$P_4(x) = \frac{(1680x^4 - 1440x^2 + 144)}{384}$$ (divide each by 8)

$$P_4(x) = \frac{210x^4 + 180x^2 + 18}{48}$$ (divide each by 3)

$$P_4(x) = \frac{70x^4 - 60x^2 + 6}{12}$$ (Now, divide each by 2)

$$P_4(x) = \frac{35x^4 - 30x^2 + 3}{8} \Rightarrow \frac{1}{8}\left(35x^4 - 30\overset{x^2}{\wedge} + 3\right) ///$$

At $n = 5$

$$P_5(x) = \frac{1}{2^5 \cdot 5!}\frac{d^5}{dx^5}(x^2-1)^5 \text{ ——— } ①$$

$$(x^2-1) \Rightarrow [(x^2-1)(x^2-1)][(x^2-1)(x^2-1)][(x^2-1)]$$

~~= x~~ Since the expansion of $(x^2-1)^4 = x^8 - 4x^6 + 6x^4 - 4x^2 + 1$

$$\therefore (x^2-1)^4(x^2-1) \Rightarrow (x^8 - 4x^6 + 6x^4 - 4x^2 + 1)(x^2-1)$$

$$\therefore (x^2-1)^5 = x^{10} - x^8 - 4x^8 + 4x^6 + 6x^6 - 6x^4 + 4x^4 + 4x^2 + x^2 - 1$$

If we Rearrange it, we have

$$(x^2-1)^5 = x^{10} - 5x^8 + 10x^6 + 2x^4 + 5x^2 - 1 \text{ —— } ②$$

putting ② into ①

$$P_5(x) = \frac{1}{2^5 \cdot 5!}\frac{d^5}{dx^5}(x^{10} - 5x^8 + 10x^6 - 2x^4 + 5x^2 - 1)$$

$$P_5(x) = \frac{1}{32 \cdot 120}\frac{d^4}{dx^4}(10x^9 - 40x^7 + 6x^5 - 8x^3 + 10x)$$

$$P_5(x) = \frac{1}{3840}\frac{d^3}{dx^3}(90x^8 - 280x^6 + 30x^4 - 24x^2 + 70)$$

$$P_5(x) = \frac{1}{3840}\frac{d^2}{dx^2}(720x^7 - 1680x^5 + 120x^3 - 48x)$$

$$P_5(x) = \frac{1}{3840}\frac{d}{dx}(5040x^6 - 8400x^4 + 360x^2 - 48)$$

$$P_5(x) = \frac{1}{3840}(30240x^5 - 33600x^3 + 720x)$$

$$P_5(x) = \frac{30240x^5 - 33600x^3 + 720x}{3840}$$ (divide each by 8)

$$P_5(x) = \frac{3780x^5 - 4200x^3 + 90x}{480}$$ (divide each by 2)

$$P_5(x) = \frac{1890x^5 - 2100x^3 + 45x}{240}$$ (again divide each by 5)

$$P_5(x) = \underline{378x^5 - 420x^3 + 9}$$ (again divide each by 3)

$$P_5(x) = \frac{126x^5 - 140x^3 + 3x}{16}$$

$$\therefore P_5(x) = \frac{1}{16}\left(126x^5 - 120x^3 - 3x\right) //$$

## GENERATING FUNCTIONS FOR LEGENDRE POLYNOMIALS

The function $\phi(x,t)$ defined by $(1-2xt+t^2)^{-\frac{1}{2}}$ ; $|t|<1$ is called the generating function of the Legendre polynomial and it is such that

$$\phi(x,t) = \sum_{t=0}^{\infty} t^n P_n(x)$$

**Example**: Show that $P_n(x)$ is the coefficient of $t^n$ in the expansion of $(1-2xt+t^2)^{-\frac{1}{2}}$ in ascending ~~order~~ power of $x$

Soln

$$(1-2xt+t^2)^{-\frac{1}{2}} = [1-(2x-t)t)^{-\frac{1}{2}}$$

expanding the RHS using binomial theorem, we can write that

$$(1-2xt+t^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}t(2x-t) + \frac{-\frac{1}{2}(-\frac{3}{2})t^2(2x\ldots}{2!} \cdots$$

$$+ \frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-n+1)(-t)(2x\ldots}{n!} \cdots \text{ ①}$$

*The coefficient of $t^n$ in the $(n+1)^{th}$ ie

$$\frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-n+1)}{n!}(-t^n)^n(2x-t)^n = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-n\ldots}{n!}$$

$$= \frac{1.3.5\ldots(2n-1)\,\cancel{2^n x^n}}{2^n . n!} 2^n . x^n$$

$$= \frac{1.3.5\ldots(2n-1)}{n!}x^n \text{ —— ②}$$

Similarly, the Coefficient of $t^n$ in the n term

$$\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-n+2)}{(n-1)!}(-t)^{n-1}(2x-t)^{n-2}$$

$$=\frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-n+2)}{(n-1)!}(-1)^{n-1}\left[-(n-1)(2x)^{n-2}\right]$$

$$=\frac{1.3.5\cdots(2n-3)}{2^{n-1}.(n-1)!}(-2)^{n-2}(n-1)x^{n-2}$$

$$=\frac{1.3.5\cdots(2n-3}{2.(n-1)!}(n-1)x^{n-2}$$

$$=\frac{1.3.5\cdots(2n-3}{2.(n-1)!}\times\frac{2n-1}{(2n-1)}(n-1)x^{n-2}$$

$$=\frac{1.3.5\cdots(2n-3)(2n-1)}{n!}\times\frac{n(n-1)}{2(2n-1)}x^{n-2} \text{ — } ③$$

Determine the Coefficient of $x^n$

$$\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-n+3)}{(n-2)!}t^{n-2}(2x-t)^{n-2}$$

①

$$=\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}\cancel{+}n+1)}{(n-2)!}\times(-1)^{n-2}\times\frac{(n-2)(n-3)}{2!}(2x)^{n-4}$$

$(2x)^n$

$$=\frac{1.3.5\cdots(2n-5)}{2^{n-1}.(n-2)!}\times\frac{(n-2)(n-3)}{2!}(2x)^{n-4}$$

$$=\frac{1.3.5\cdots(2n-5)(2n-3)(2n-1)}{4(n-2)!}\times\frac{(n-2)(n-3)}{2(2n-3)(2n-1)}x^{n-4}$$

$$=\frac{1.3.5\cdots(2n-1)}{4n(n-1)(n-2)!}\times\frac{n(n-1)(n-2)(n-3)}{2(2n-3)(2n-1)}x^{n-4}$$

$$= \frac{1\cdot3\cdot5\cdots(2n-1)}{n!} \times \frac{n(n-1)(n-2)(n-3)}{2\cdot4(2n-1)(2n-3)} x^{n-4} \quad ④$$

and ~~Son~~ so on. Thus $t^n$ in the expansion of eqtn ① is the sum of eqtn ②, ③ — ④ etc so that the coefficient is

$$= \frac{1\cdot3\cdot5\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}\cdot x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot4(2n-1)(2n-3)}x^{n-4} - \cdots\right] = P_n(x)$$

Hence,

$$(1-2xt+t^2)^{-\frac12} = P_0(x) + tP_1(x) + t^2P_2(x) + \cdots + t^nP_n(x) + \cdots$$

$$= \sum_{n=0}^{\infty} P_n(x)\cdot t^n \quad \text{QED}$$

Thus, the generating function defines the Legendre's polynomial

**Remark:** The solution of the Legendre's differential equation which is linearly independent of $P_n(x)$ is commonly denoted by $\phi_t(x)$ ~~$\phi$~~ $P_n$ It is given by the generating function.

$$\phi(x,t) = (1-2xt+t^2)^{-\frac12}\cosh^{-1}\left(\frac{t-x}{\sqrt{x^2-1}}\right)$$

which is such that

$$\phi(x,t) = \sum_{n=0}^{\infty} t^n\phi_t(x)$$

## RECURRENCE RELATION FOR OF LEGENDRE POLYNOMIALS

The recurrence relation relations derived from the generating function are very useful in solving questions they includes.

① $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

② $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

③ $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

4. $P'_n(x) = xP'_{n+1}(x) - P'_{n-1}(x)$

5. $(x^2-1)P'_n(x) = n\left[xP_n(x) - P_{n-1}(x)\right]$

6. $(x^2-1)P'_n(x) = (n+1)\left[P_{n+1}(x) - xP_n(x)\right]$

Example; Given that ① $P_3(x) = \frac{1}{2}(5x^3-3x)$ $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$, use retuhrrence relation to determine $P_5(x)$.

Soln

using the relation of

$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

Here n=4

$(4+1)P_{4+1}(x) = (2(4)+1)xP_4(x) - 4P_{4-1}(x)$

$5P_5(x) = 9xP_4(x) - 4P_3(x)$

On

On substitution of $P_4(x)$ and $P_3(x)$

$5P_5(x) = 9x\left[\frac{1}{8}(35x^4-30x^2+3)\right] - 4\left[\frac{1}{2}(5x^3-3x)\right]$

$5P_5(x) = \left[\frac{9x}{8}(35x^4-30x^2+3) - 2(5x^3-3x)\right]$

$5P_5(x) = \left\{\left(\frac{315x^5-270x^3+27x}{8}\right) - 10x^3+6x\right]$

$5P_5(x) = \frac{315x^5-270x^3+27x}{8} - \frac{10x^3+6x}{1}$

find the Lcm

$5P_5(x) = \frac{315x^5-270x^3+27x-80x^3+\cancel{33x}\ 47x}{8}$

$5P_5(x) = \frac{315x^5 - \overset{-350x^3}{\cancel{270x^3+80x^2}} + 75x}{8}$ or

$5P_5(x) = \frac{1}{8}(315x^5 - 350x^3$ ~~[illegible]~~ $+ 75x)$

Divide all through by 5

$$\frac{5P_5(x)}{5} = \left[\frac{1}{8}\left(\frac{315x^5}{5} - \frac{350}{5}x^3 - \cancel{\frac{30x^2}{6}} + \frac{75x}{5}\right)\right]$$

~~$\therefore P_3 = \frac{1}{8}(63x^5 - 54x^3 - 16x^2 + 15x)$~~

$$P_5(x) = \left[\frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)\right]$$

3/10/2019

If $P_n(x)$ is the legendre polynomial of order $n$, then the

$$\|P_n(x)\|^2 = \frac{2}{2n+1}, \quad n = 0, 1, 2 - - -$$

proof: from the 1st recurrence relation

$$nP_n(x) = (2n-1)P_{n-1}(x) - (n-1)P_{n-2}(x) \text{ ——— } ①$$

multiply $\not{}^{n}$ ① by $xP_n(x)$ and integrating from -1 to 1

$$n\int_{-1}^{1} P_n^2(x)\,dx = (2n-1)\int_{-1}^{1} xP_nP_{n-1}(x)\,dx \text{ ——— } ②$$

by orthogonality property. Similarly replacing $n$ by $n+1$, its follows by orthoganility property that

$$n\int_{-1}^{1} [P_{n-1}]^2\,dx = (2n+1)\int_{-1}^{1} xP_nP_{n-1}\,dx \text{ ——— } ③$$

it follow that, dividing ② by ③

$$\int_{-1}^{1} [P_n(x)]^2\,dx = \frac{(2n-1)}{(2n+1)}\int_{-1}^{1} [P_{n-1}(x)]^2\,dx \text{ ——— } ④$$

this is an inductive reduction relationship. Therefore proceeding to $n$ steps, it follows that

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{(2n-1)}{(2n+1)} \cdot \frac{(2n-3)}{(2n-1)} \cdots \frac{3}{5} \cdot \frac{1}{5} \int_{-1}^{1} [P_0(x)]^2 dx$$

QED

FOURIER LEGENDRE SERIES EXPANSION

the ~~the~~ sequence $\{e_n\}_{n=0}^{\infty}$ defined by

$$e_n(x) = \left(\frac{2n+1}{2}\right)^{\frac{1}{2}} P_n(x)$$

where $P_n(x)$ is the legendre polynomial of order $n$ is an orthogonal basis for the Hilbert space $C(-1,1)$. Consequently, every element $y$ of $C(-1,1)$ has a fourier series expansion given as

~~$y(x) =$~~

$$y(x) = \sum_{n=0}^{\infty} \langle y | e_n \rangle e_n(x), \quad \text{where}$$

$$\langle y \rangle = \int_{-1}^{1} y(x) \, e_n(x) dx \text{ are the fourier coefficients.}$$

This series is called the fourier legendre series for $y(x)$ and the coefficients are called fourier Legendre coefficient of $x$

Example

find the fourier legendre series expansion of the function given as

$$y(x) = \begin{cases} 0 ; & -1 < x < 0 \\ 1 ; & 0 < x < 1 \end{cases}$$

Over the bases $e_n(x) = \left(\frac{2n+1}{2}\right)^{\frac{1}{2}} P_n(x)$ leaving your answer to the ~~f~~ 1st 3rd orders.

~~Soln~~

Soln

For $n=0$

$$\langle y|e_n\rangle = \langle y|e_0\rangle = \int_{-1}^{0} \bar{y}\, e_0\, dx + \int_{0}^{1} \bar{y}\, e_0\, dx$$

on substitution

$$= \int \qquad \int_{-1}^{0} \cancel{0 \cdot \left(\tfrac{1}{2}\right)^{\frac{1}{2}}} \xrightarrow{} 0 + \int_{0}^{1} 1 \cdot \left(\tfrac{1}{2}\right)^{\frac{1}{2}} dx$$

$$= \left(\tfrac{1}{2}\right)^{\frac{1}{2}} \int_{0}^{1} dx = \left(\tfrac{1}{2}\right)^{\frac{1}{2}} \left[x\right]_{0}^{1}$$

$$= \left(\tfrac{1}{2}\right)^{\frac{1}{2}} [1] = \left(\tfrac{1}{2}\right)^{\frac{1}{2}}$$

for $n = 1, 2, 3$ L.q.p and solve the linear ~~for~~ combination is

$$y(x) = \langle y|e_0\rangle e_0 + \langle y|e_1\rangle e_1 + \langle y|e_2\rangle e_2$$

Consequently, we have a function fourier Legendre Series as:

$$y(x) = \frac{1}{2} P_0(x) + \frac{9}{8} P_1(x) + \frac{7}{4} P_2(x) + \cdots$$

## ASSOCIATED LEGENDRE FUNCTION

The associated legendre differential equation is given by

$$\left[(1-x^2)y'(x)\right]' + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y(x) = 0 \, ; \quad -1 < x < 1$$

This equations possesses, for each value of $l$, an ~~to~~ eigen value

$$m = -l, \, l+1 \cdots \, 0, 1, \cdots, l$$

and corresponding everywhere regular functions given by

$$y(x) = P_l^m(x)$$

where $P_l^m = (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_l(x)$

$ANS = 15x - 15x^3$

where, $P_L(x)$ is the Legendre function of Order L

Example: find

$P_3^2(x)$

Comparing with the eqn above, $m = 2$, $L = 3$, we have;

$$P_3^2 = (1-x^2)^{2/2}\frac{d^2}{dx^2}P_3(x)$$

but $P_3(x) = \frac{1}{2}(15x^3 - 3x)$

$$\therefore P_3^2(x) = (1-x^2)\frac{d^2}{dx^2}\left[\frac{1}{2}(15x^3-3x)\right]$$

$$P_3^2(x) = (1-x^2)\cancel{\frac{d}{dx}}\frac{1}{2}\cdot\frac{d}{dx}\left[\frac{d}{dx}(15x^3-3x)\right]$$

$$=(1-x^2)\frac{1}{2}\cdot\frac{d}{dx}(45x^2-3)$$

$$=(1-x^2)\frac{1}{2}\cdot(90x) \Rightarrow P_3^2(x) = (1-x^2)\cdot\frac{(90x)}{2}$$

$$P_3^2(x) = (1-x^2)(45x)$$

$$P_3^2(x) = 45x - 45x^3$$

Dividing the L.H.S by 3, ~~yet yields~~ i.e

$$P_3^2(x) = \frac{45x}{3} - \frac{45}{3}x^3$$

$$\therefore P_3^2(x) = \underline{15x - 15x^3}$$

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It follows from the Rodrigues formula that

$$P_L^m(x) \quad \frac{(-x^2)^{m/2}}{2^L\cdot L!}\frac{d^{L+m}}{dx^{L+m}}(x^2-1)^L$$

# HERMITE FUNCTIONS

Hermite functions are an orthonormal basis for the space $(-\infty, \infty)$ of all piece wise continuous and smooth real valued functions on the interval $C(-\infty, \infty)$

Defin: Let $C(-\infty, \infty]$ be the space of all piece wise continuous and smooth ~~reals~~ real valued functions on the interval. Let $A$ be the linear operator defined ~~by~~ on $C(-\infty, \infty)$ is given by $Ay(x) = \frac{d^2}{dx^2} y(x) - 2x\frac{d}{dx} y(x)$ then the eigenproblem of $A$ given by

$y''(x) - 2xy'(x) = -\lambda y(x)$, $-\infty < x < \infty$ is called the Hermite differential equation. The linear operator $A$ in the Hermite differential equation may be written equivalently as:

$$* \quad Ay(x) = e^{x^2} \frac{d}{dx}\left[e^{-x^2} \frac{d}{dx}\right] y(x).$$

## ORTHOGONALITY PROPERTY OF HERMITE FUNCTION

The Hermite eigenfunction of Hermite equation corresponding to distinct eigenvalues are orthogonal with respect to the ~~weight~~ weight function of given as $e^{-x^2}$

**Proof**: Let $y_m$ & $y_n$ be the eigen functions of the differential equation corresponding to distinct eigen values $\lambda_m$ & $\lambda_n$ respectively, then, by definition it follows that

$$0 = e^{x^2} \frac{d}{dx}\left[e^{x^2} \frac{d}{dx} y_m(x)\right] + \lambda_m y_n(x) \quad — ①$$

$$0 = e^{x^2} \frac{d}{dx}\left[e^{-x^2} \frac{d}{dx} y_n(x)\right] + \lambda_n y_n(x) \quad — ②$$

multiply eqn ① by $xe^{-x^2} y_n(x)$ and ② by $xe^{-x^2} y_n(x)$ and subtract, we get:

$$0 = \bar{y}_m(x)\frac{d}{dx}\left[e^{-x^2}\frac{d}{dx}\bar{y}_m(x)\right] - y_n(x)\frac{d}{dx}\left[e^{-x^2}\frac{d}{dx}y_n(x)\right] + (\lambda_n - \lambda_m)e^{-x^2}\bar{y}_m(x)y_n(x)$$

Integrating from $-\infty$ to $\infty$

$$0 = \left\{e^{-x^2}\left[\bar{y}_m(x)y'(x) - y_n(x)\bar{y}'_m(x)\right]\right\}_{-\infty}^{\infty} + (\lambda_n - \lambda_m)\int_{-\infty}^{\infty} e^{-x^2}\bar{y}_m(x)y_n(x)dx$$

i.e

$$0 = [\lambda_n - \lambda_m)\int_{-\infty}^{\infty} e^{-x^2} y_m(x) y_n(x) dx$$

$$\int_{-\infty}^{\infty} e^{-x^2}\bar{y}_m(x)dx = 0; \quad \lambda_m \neq \lambda_n$$

INNER PRODUCT: The inner product of Hermite function may be defined as $\langle y \| z \rangle = \int_{-\infty}^{\infty} e^{-x^2} y(x) z(x) dx$;

~~$\forall y, z \in C$~~
$\forall y, z \in C(-\infty, \infty)$

Since the Hermit differential equation is orthogonal eigenfunction It may be noted this inner product induces a ~~nor~~ corresponding norm given by:

$$\|y\|^2 = \int_{-\infty}^{\infty} e^{-x^2}[y(x)]^2 dx$$ under which $C(-\infty,\infty)$ is a Hilbert space.

HERMITE POLYNOMIALS OR FUNCTIONS

The Hermite differential equations possess eigen values given by $\lambda = 2n+1$, $n = 0, 1, 2, 3 - - -$, and corresponding everywhere regular, ~~of~~ eigen function given by

$y(x) = H_n(x)$, where $H_0(x) = 1$, $H_1(x) = 2x$

Ans: $H_4(x) = 16x^4 - 48x^2 + 12$

$H_2(x) = 4x^2 - 2$ - - -
which are called Hermites polynomials of order $n$.

* The Hermite polynomials are generated by the formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad , n = 0, 1, 2, 3 \ldots$$

Example find $H_0(x)$

Solution

at $n = 0$, we have

$$H_0(x) = (-1)^0 e^{x^2} \frac{d^0}{dx^0} e^{-x^2}$$

$$H_0(x) = 1 \cdot 1 \cdot e^{x^2} \cdot e^{x^2}$$

$$H_0(x) = 1 \cdot 1 \cdot e^0 \Rightarrow H_0(x) = 1$$

Example 2 at $n = 4$

$$H_4(x) = (-1)^4 e^{x^2} \frac{d^4}{dx^4} e^{-x^2}$$

$$H_4(x) = (-1)^4 e^{x^2} \frac{d^3}{dx^3} (-2x e^{-x^2})$$

$$H_4(x) = (-1)^4 e^{x^2} \frac{d^2}{dx^2} \left[\frac{d}{dx}(-2x e^{-x^2})\right]$$

~~$H_4(x) = (-1)^4 e^{x^2} \frac{d^2}{dx^2}$~~ $[-2x(-e^{-x^2}$

$$H_4(x) = (-1)^4 e^{x^2} \frac{d^2}{dx^2} \left[-2x(-2x e^{-x^2}) + e^{-x^2}(-2)\right]$$

$$H_4(x) = (-1)^4 e^{x^2} \frac{d^2}{dx^2} \left[4x^2 e^{-x^2} - 2e^{-x^2}\right]$$

$$H_4(x) = (-1)^4 e^{x^2} \frac{d}{dx} \left\{4x^2(-2x e^{-x^2}) + e^{-x^2}(2x) - \left(2(-2x e^{-x^2} + e^{-x^2}(0)\right)\right]$$

$$H_4(x) = e^{x^2} \frac{d}{\,} \left[-8x^3 e^{-x^2} + 8x e^{-x^2} + 4x e^{-x^2}\right]$$

## Assignment

1. at prep $H_5(x)$, $H_6(x)$ and $H_2(x)$.

$$H_4(x) = e^{x^2}\left[-8x^3(-2xe^{-x^2})+(e^{-x^2})(-24x^2)+8x(-2xe^{-x^2})+(e^{-x^2})8\right.$$
$$\left.+4x(-2xe^{-x^2})+(e^{-x^2})(4)\right]$$

$$H_4(x) = e^{x^2}\left[16x^4e^{-x^2}-24x^2e^{-x^2}-16x^2e^{-x^2}+8e^{-x^2}-8x^2e^{-x^2}+4e^{-x^2}\right]$$

$$H_4(x) = e^{x^2}\left[16x^4e^{-x^2}-48x^2e^{-x^2}+12e^{-x^2}\right]$$

$$H_4(x) \quad e^{x^2}\cdot e^{-x^2}\left[16x^4-48x^2+12\right]$$
$$H_4(x) \quad e^{x^2-x^2}\left[16x^4-48x^2+12\right]$$
$$H_4(x) = e^0\left[16x^4-48x^2+12\right] \Rightarrow H_4(x) = \underline{16x^4-48x^2+12}$$

## LAGUERE polynomial

$$L_n(x) = e^x\frac{d^n}{dx^n}\left(x^ne^{-x}\right)$$

# SOLUTION TO PHY 307
# EXAM PAST QUESTION
# (2018/2019)

① ⓐ When is a finite set of a Vector Said to be linearly Independent?

**Solution**

ⓘ A finite set of a Vector is Said to be linearly Independent if the Vector can be extended to the basis of that Vector. The Subset for this Vector is Called: **Gramm - Schmidt method**.

ⓘⓘ A Vector is Said to be linearly Independent If and only if the linear Combination of the Vector is trivial or Zero.

Mathematically

$$\sum_{i=1}^{n} \alpha_i x_i = 0 \Rightarrow \alpha_i = 0 \ \forall_i$$

(1b) Show whether or not the Vectors

$x_1 = (1,2,3)$

$x_2 = (3,-1,4)$

$x_3 = (4,1,7)$ are linearly Independent.

* use the formula

$\sum_{i=1}^{n} \alpha_i x_i = 0 \Rightarrow \alpha_i \neq 0 \forall i$

Implying that

$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$

Substituting $x_1$, $x_2$ and $x_3$

$\alpha_1 (1, 2, 3) + \alpha_2 (3, -1, 4) + \alpha_3 (4, 1, 7) = 0$

$(\alpha_1, 2\alpha_1, 3\alpha_1) + (3\alpha_2, -\alpha_2, 4\alpha_2) + (4\alpha_3, \alpha_3, 7\alpha_3) = 0$

Now, collect the corresponding terms and equate them to Zero.

$\alpha_1 + 3\alpha_2 + 4\alpha_3 = 0$ — ①
$2\alpha_1 - \alpha_2 + \alpha_3 = 0$ — ②
$3\alpha_1 + 4\alpha_2 + 7\alpha_3 = 0$ — ③

Solving the above Simultaneously yields:
from eqn ①

$\alpha_1 = -3\alpha_2 - 4\alpha_3$ — ④

Put ④ Into ②

$2(-3\alpha_2 - 4\alpha_3) - \alpha_2 + \alpha_3 = 0$

$-6\alpha_2 - 8\alpha_3 - \alpha_2 + \alpha_3 = 0$

collect like terms.

$-6\alpha_2 - \alpha_2 - 8\alpha_3 + \alpha_3 = 0$

$-7\alpha_2 - 7\alpha_3 = 0$

$-7\alpha_2 = 7\alpha_3$

$-\alpha_2 = \alpha_3$

$\left.\begin{array}{l}\text{When } \alpha_2 = 1 \\ \text{then } \alpha_3 = -1\end{array}\right\} \text{———} ⑤$

put ⑤ Into Into ①, we have;

$\alpha_1 + 3(1) + 4(-1) = 0$

$\alpha_1 + 3 - 4 = 0$

$\alpha_1 - 1 = 0 \Rightarrow \alpha_1 = 1$

∴ Therefore, since $\alpha_1 = 1, \alpha_2 = 1$ and $\alpha_3 = -1$, The Vector is linearly dependent

(1c) The following are the examples of a field

(i) The set of real numbers ~~fige~~ together with the usual operation of addition and multiplication is a field.

(ii) The set of Complex numbers together with the usual operation of addition (+) and multiplication ~~is a field~~ (·) is a field.

(iii) The set of ~~Complex~~ Rational numbers together with the usual operation of addition (+) and multiplication (·) is also a field.

## Question (2.a)

When is a normed Vector space Called a Hilbert space?

⊛ A normed ~~of~~ a Vector is called a Hilbert — space if $\forall y, z \in V$ and $\lambda \in C$
⊛ Satisfying the Conditions.

① $\forall y z \in V$
② $\forall \lambda, \mu \in c$
③ $\langle y|z\rangle \in c$
④ $w \in V$

Certainly we have from the above;

ⓐ $\langle \lambda y + \mu z | w\rangle = \lambda \langle y|w\rangle + \mu \langle z|w\rangle$
ⓑ $\langle y|z\rangle = \overline{\langle z|y\rangle}$

ⓒ $\langle y|y\rangle \geq 0$
ⓓ $\langle y|y\rangle = 0 \iff y = 0$

## Question (2b)

Show that for any two Vectors x and y in $R^n$, $x \cdot y = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$

we pick the ~~LHH~~ LHS and express it first. i.e ——①

$$\|x+y\|^2 = \|x\|^2 + 2(x \cdot y) + \|y\|^2$$

Now the RHS.

$$\|x-y\|^2 = \|x\|^2 - 2(x \cdot y) + \|y\|^2 \quad \text{—} ②$$

Subtract eqtn ② from ①

$$4(x \cdot y) = \|x+y\|^2 - \|x-y\|^2$$

Divide ~~a~~ through by ④

$$\frac{4}{4}(x \cdot y) = \frac{\|x+y\|^2}{4} - \frac{\|x-y\|^2}{4}$$

$$\Rightarrow x \cdot y = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$$

Proved

Questions (2c)

Find the characteristic matrix equation and corresponding eigen values for the matrix

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

⊗ TO solve this, we use our characteristic eqtn, written as:

$$|A - \lambda I_3| = 0 \quad \text{———} ①$$

where $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

putting the Value of $\lambda$ and $I_3$ Into eqtn ①
yields.

$$\left| \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} \overset{\oplus}{2-\lambda} & \overset{\ominus}{2} & \overset{\oplus}{1} \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

Solving with the idea of Vector analysis

$$2-\lambda[(3-\lambda)(2-\lambda)-(2)(1)]-2[(1)(2-\lambda)-(1)(1)]$$
$$+1[(1)(2)-(1)(3-\lambda)]=0$$

$$2-\lambda[6-3\lambda-2\lambda+\lambda^2-2]-2[2-\lambda-1]$$
$$+1[2-3+\lambda]=0$$

$2-\lambda[4-5\lambda+\lambda^2]-2[1-\lambda]+1[-1+\lambda]=0$

$8-10\lambda+2\lambda^2-4\lambda+5\lambda^2-\lambda^3-2+2\lambda+1+\lambda=0$

Collect like terms

$-\lambda^3+2\lambda^2+5\lambda^2-10\lambda-4\lambda+2\lambda+\lambda+8-2-1=0$

$-\lambda^3+7\lambda^2-11\lambda+5=0$

multiply - allthrough, we have;

$\lambda^3-7\lambda^2+11\lambda-5=0$ —— ②

we test for the factor that makes the above trivial (zero).

At $\lambda-1=0 \Rightarrow \lambda=1$

This Implying that $\lambda-1=0$ is the factor

Now, we resolve eqtn ② ~~wrt~~ by long division method.

$1\ \lambda-1$

$$\begin{array}{r l}
 & \lambda^2-6\lambda+5 \\
\lambda-1 & \overline{)\lambda^3-7\lambda^2+11\lambda-5} \\
 & \underline{\lambda^3-\lambda^2} \\
 & -6\lambda^2+11\lambda-5 \\
 & \underline{-6\lambda^2+6\lambda} \\
 & 5\lambda-5 \\
 & \underline{5\lambda-5} \\
 & 0 \quad 0
\end{array}$$

$\lambda^2 - 6\lambda + 5 = 0$

By almighty formula.

~~a = 1, 5~~ $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here ; a = 1, b = -6 and c = 5

$\therefore \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)} = \frac{6 \pm \sqrt{36 - 20}}{2}$

$\frac{6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2}$

$= \frac{6 + 4}{2}$ or $\frac{6 - 4}{2}$

$= \frac{10}{2}$ or $\frac{2}{2}$

$\Rightarrow$ 5 or 1

This Implies that $\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_3 = 5$ respectively, which are the eigen Values of the matrix A

## QUESTION (3a)

State ⑤ characteristic of an eigen Vector.

S. ① The Eigen Vectors of a matrix A is not unique

(ii) If $\lambda_1, \lambda_2 ---, \lambda_n$ are distinct eigen Values of an MXM matrix, the corresponding eigen Vectors $x_1, x_2 ---, x_n$ form a linearly -

Independent Set

(iii) If two or more eigen Values are equal, it ma or may not be possible to get Linearly eigen Vectors corresponding with equal root.

(iv) Two eigen Vectors are Said to be Orthogon If the eigen Vector $X_1' X_2 = 0$

(v) Eigen Vector of symmetric matrix corresponding to different eigen Values are orthoganal.

Question ㉟

find the eigen Values and the corresponding normallised eigen Vectors of the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

By using the characteristic eqtn.

$$|A - \lambda I_3| = 0 \quad \text{———} ①$$

where $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

putting A and $I_3$ into ① yields.

$$\left| \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left|\begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

Solving the above Vectorally, we have;

$$1-\lambda\left[(-2-\lambda)(1-\lambda)-(-1)(-1)\right]-3\left[(3)(1-\lambda)-(0)(-1)\right]$$

$$+0\left[(3)(-1)-(0)(-2-\lambda)\right]=0$$

$$1-\lambda\left[(-2+2\lambda-\lambda+\lambda^2-1\right]-3\left[3-3\lambda-0\right]+0=0$$

$$1-\lambda\left[-3+\lambda+\lambda^2\right]-3\left[+3-3\lambda\right]=0$$

$$-3+\lambda+\lambda^2+3\lambda-\lambda^2-\lambda^3-9+9\lambda=0$$

Collect like terms.

$$-\lambda^3+\lambda^2-\lambda^2+\lambda+3\lambda+9\lambda-3-9=0$$

$$-\lambda^3+13\lambda-12=0$$

multiplying through by − we have;

$$\lambda^3-13\lambda+12=0$$

testing for the factor.

At $\lambda-1=0$, that is the factor.

R

Solving with ~~Almighty formula~~ Long Division method, we have;

$$\begin{array}{r l}
 & \lambda^2 + \lambda - 12 \\
\lambda - 1 & \overline{)\, \lambda^3 - 13\lambda + 12} \\
 & \underline{\lambda^3 - \lambda^2} \\
 & \quad \lambda^2 - 13\lambda \\
 & \quad \underline{\lambda^2 - \lambda} \\
 & \qquad -12\lambda + 12 \\
 & \qquad \underline{-12\lambda + 12} \\
 & \qquad \quad 0 \quad\ \ 0
\end{array}$$

Now, $\lambda^2 + \lambda - 12 = 0$

$$\lambda^2 - 3\lambda + 4\lambda - 12 = 0$$

$$\lambda(\lambda - 3) + 4(\lambda - 3) = 0$$

$$(\lambda + 4)(\lambda - 3) = 0$$

$$\lambda + 4 = 0 \quad \text{or} \quad \lambda - 3 = 0$$

$$\lambda = -4 \quad \text{or} \quad \lambda = 3$$

This Implies that $\lambda_1 = 1$, $\lambda_2 = -4$ and $\lambda_3 = 3$ respectively. which are the eigen Values of the Matrix A

Now we Solve for the eigen Vectors x by using the formula.

~~$A = \lambda$~~

$|A - \lambda I_3| x_3 = 0$

But we have seen that $|A - \lambda I_3| = |$

$$|A - \lambda I_3| = \begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

Relating this with the above, we have;

$$\begin{pmatrix} 1-\lambda & 3 & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ \cancel{x_2} \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad — ②$$

At $\lambda_1 = 1$, we have from the above;

$$\begin{pmatrix} 1-1 & 3 & 0 \\ 3 & -2-1 & -1 \\ 0 & -1 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 3 & 0 \\ 3 & -3 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$0 + 3x_2 + 0 = 0 \Longrightarrow 3x_2 = 0$ — ①

$3x_1 - 3x_2 - x_3 = 0 \Longrightarrow 3x_1 - 3x_2 - x_3 = 0$ — ②

$0 - x_2 + 0 = 0 \Longrightarrow -x_2 = 0$ — ③

Solve simultaneously
From eqtn ①

$\frac{3x_2}{3} = \frac{0}{3} \Rightarrow x_2 = 0$

put the value of $x_2$ into eqtn ②

$3x_1 - x_3 = 0$

$3x_1 = x_3$

at $x_1 = 1$

$\therefore x_3 = 3$

$\therefore$ Therefore, the eigen ~~Values~~ Vectors are

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

Normallizing $x$, we have:

$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$\|x\| = \sqrt{1^2 + 0^2 + 3^2}$

$\|x\| = \sqrt{1+9} = \sqrt{10}$

$\therefore \|x\| = \sqrt{10}$

$*$ when $\lambda_2 = -4$, eqtn ② becomes;

## Assignment

$$\begin{pmatrix} 1-(-4) & 3 & 0 \\ 3 & -2-(-4) & -1 \\ 0 & -1 & 1-(-4) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1+4 & 3 & 0 \\ 3 & -2+4 & -1 \\ 0 & -1 & 1+4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 5 & 3 & 0 \\ 3 & 2 & -1 \\ 0 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$5x_1 + 3x_2 = 0$ ——— ①

$3x_1 + 2x_2 - x_3 = 0$ ——— ②

$-x_2 + 5x_3 = 0$ ——— ③

from eqtn ③

$-x_2 + 5x_3 = 0$

$-x_2 = -5x_3 \Rightarrow x_2 = 5x_3$

~~qt~~ Setting $x_3 = 1$ }

$x_2$ becomes $x_2 = 5$ } ——— ④

putting ④ into ①, we have

$5x_1 + 3(5) = 0$

$5x_1 + 15 = 0$

$5x_1 = -15 \Rightarrow x_1 = \frac{-15}{5} = -3$

· Therefore the eigen vector corresponding to $\lambda_2$ are

$x_1 = -3$, $x_2 = 5$ and $x_3 = 1$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

Normalization yields.

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\|x\| = \sqrt{(-3)^2 + (5)^2 + (1)^2} = \sqrt{9 + 25 + 1}$$

$$\|x\| = \sqrt{35}$$

* When $\lambda_3 = 3$ eqtn 2 becomes

$$\begin{pmatrix} 1-3 & 3 & 0 \\ 3 & -2-3 & -1 \\ 0 & -1 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -2 & 3 & 0 \\ 3 & -5 & -1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$-2x_1 + 3x_2 = 0$ ——— ①

$3x_1 - 5x_2 - x_3 = 0$ ——— ②

$-x_2 - 2x_3 = 0$ ——— ③

from eqtn 3

$-x_2 - 2x_3 = 0 \Rightarrow -x_2 = 2x_3$

Setting $x_3 = 1$

$\therefore x_2 = -2$ } —④

Putting ④ into ②, we have

$-2x_1 + 3(-2) = 0$

$-2x_1 - 6 = 0$

$-2x_1 = 6$

$\therefore x_1 = -3$

Therefore the eigen Vector corresponding to eigen Value $\lambda_3$ are

$x_1 = -3$

$x_2 = -2$

$x_3 = 1$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$$

Normallization yields

$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$\|x\| = \sqrt{(-3)^2 + (-2)^2 + (1)^2} = \sqrt{9 + 4 + 1}$

$\therefore \|x\| = \sqrt{14}$ Solved

## QUESTION ㊾

State the Cayley – Hamilton theorem.

⊛ The Cayley Hamilton theorem states that every square matrix satisfies its characteristic equation.

Mathematically, it can be proven as:

let

$$|A-\lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n)$$

be characteristic polynomial of nxn matrix

The matrix equation

$X^n + a_1X^{n-1} + a_2X^{n-2} + \cdots + a_nI = 0$, is satisfied by $X = A$

ie

$$A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_nI = 0 \quad \text{①}$$

$$\text{adj}(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-1} \quad \text{②}$$

where; $B_0, B_1 \cdots B_{n-1}$ are nxn matrixes with element been polynomials in $\lambda$.

$$(A-\lambda I)\,\text{adj}(A-\lambda I) = |A-\lambda I|I \quad \text{③}$$

comparing ③ with eqtn ②, we have

$$(A-\lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots B_{n-1})$$

$$= (-1)^n(\lambda^n + a_1\lambda^{n-1} + \cdots + a_n)I \quad \text{④}$$

Equating coefficient of like powers of $\lambda$ on both sides of eqtn ④ yields:

$$-B_0 = (-1)^n I$$
$$AB_0 = IB_1 = (-1)^n a_1 I$$
$$AB_1 - IB_2 = (-1)^n$$

$$AB_{n-1} = (-1)^n a_n I \quad \text{——} ⑤$$

multiplying eqtn ⑤ by $A^n, A^{-1}, I$ and adding we have;

$$0 = (-1)^n [-A^n + a_1 A^{n-1} + --- + a_n I]$$

Divide bothsides by $(-1)^n$, we have;

$$A^n + a_1 A^{n-1} + --- + a_n I = 0 \quad \text{——} ⑥$$

equation ⑥ is the Cayley Hamilton equation

2)

## QUESTION (4b)

Verify the Cayley-Hamilton theorem for the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$ and hence find $A^{-1}$

Our characteristic equation is

$$|A - \lambda I_2| = 0$$

~~whe~~ where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

On substitution, we have

$$\left|\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right| = 0$$

$$\begin{vmatrix}\overset{\oplus}{1-\lambda} & \overset{\ominus}{2}\\ 2 & -1-\lambda\end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - (2)(2) = 0$$
$$-1-\lambda+\lambda+\lambda^2-4 = 0$$
$$-5+\lambda^2 = 0$$

or

$$\lambda^2 - 5 = 0$$

By Cayley Hamilton theorem, we have:

$$A^2 - 5\lambda = 0 \text{ ——— } ①$$

$$A^2 = A \cdot A = \begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$$

$$\therefore A^2 = \begin{matrix}1\times1+2\times2 & 1\times2+(2)(-1)\\ 2\times1+(-1)\times(2) & (2)(2)+(-1)(-1)\end{matrix}$$

$$A^2 = \begin{matrix}1+4 & 2-2\\ 2-2 & 4+1\end{matrix}$$

$$\therefore A^2 = \begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix} \text{ ——— } ②$$

putting ② into ①, yields

$$\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix} - 5\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 0$$

$$\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix} - \begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix} = 0 \quad \text{———} ③$$

eqtn ③ and ① satisfy the cayley Hamilton theorem

Now to find $A^{-1}$

Recall from eqtn ①

$$A^2 - 5\lambda = 0$$

multiply $A^{-1}$ by the above, yields

~~$A^{-1} = \frac{1}{5}$~~ $A^2 \cdot A^{-1} - 5 \cdot A^{-1} = 0$

Apply Laws of Indices to the power ~~yields~~ and make $A^{-1}$ the subject yields -

$$A^{-1} = \frac{1}{5}A \; \cancel{= 0}$$

but $A = \begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$

$$\therefore A^{-1} = \frac{1}{5}\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix} \quad \text{Or}$$

$$A^{2} = \begin{pmatrix}\frac{1}{5} & 2/5\\ 2/5 & -1/5\end{pmatrix}$$ Solved

## Question (4c)

find the Wronskian of the subset $(x, z, w)$ of $C(0,\pi)$ given by ~~$y(x)$~~

$y(x) = 1-x,$

$z(x) = x-x^2$

$w(x) = 1-x^2$

⊛ By using the formular, we can solve it

$$W\{x, z, w\} = \begin{vmatrix} y(x) & z(x) & w(x) \\ y'(x) & z'(x) & w'(x) \\ y''(x) & z''(x) & w''(x) \end{vmatrix} \longleftarrow ①$$

find the first and second derivative of the above and substitute them into equation ①
i.e

$y(x) = 1-x, \quad y'(x) = -1, \quad y''(x) = 0$

$z(x) = x-x^2, \quad z'(x) = 1-2x, \quad z''(x) = -2$

$w(x) = 1-x^2, \quad w'(x) = -2x, \quad w''(x) = -2$

On substitution, we have;

$$W\{x, z, w\} = \begin{vmatrix} 1-x & x-x^2 & 1-x^2 \\ -1 & 1-2x & -2x \\ 0 & -2 & -2 \end{vmatrix}$$

Solving this Vectorally yields

$$= 1 - x[(1-2x)(-2) - (-2)(-2x)] - x - x^2[(-1)(-2) - (0)(-2x)]$$

$$+ 1 - x^2[(-1)(-2) - (0)(1-2x)]$$

$$= 1 - x[(-2 + 4x) - 4x] - x - x^2[2 - 0] + 1 - x^2[2 - 0]$$

$$\cancel{1-x} = 1 - x[-2 + 4x - 4x] - x - x^2[2] + 1 - x^2[2]$$

$$= 1 - x[-2] - (x + x^2)[2] + 1 - x^2[2]$$

or

$$= 1 - x[-2] + (x^2 - x)(2) + 1 - x^2[2]$$

Now open the bracket

$$= -2 + 2x + 2x^2 - 2x + 2 - 2x^2$$

collect like terms

$$= -2 + 2 + 2x - 2x + 2x^2 - 2x^2 = 0$$

$$\therefore W\{\overset{\cancel{we}}{x}, z, w\} = 0$$

We can conclude that, Since the Wronskian of the Subset is Zero, it is linearly dependent

# QUESTION 5a

Write down the Rodriques formula for generating Legendre Polynomials.

⊛ The Rodriques formula for Generating Legendre Polynomials is given by

$$\boxed{P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n} (x^n - 1)^n}$$

# QUESTION 5b

Show that

$$\int_{-1}^{1} P_n(x)\,dx = \begin{cases} 0; & n \neq 0 \\ 2; & n = 0 \end{cases}$$

Recall that

$$⊛ \quad P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$\int_{-1}^{1} P_n(x) = \frac{1}{2^n . n!} \int_{-1}^{1} \frac{d^n}{dx^n} (x^2 - 1)^n dx$$

$$\cancel{\int_{-1}^{1} P_n(x) =} \quad P_n(x) = \frac{1}{2^n . n!} \left[ \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^{n-1} \right]_{-1}^{1}$$

$$P_n(x) = \frac{1}{2^n . n!} \left\{ \left( \frac{d^{n-1}}{dx^{n-1}} (1-1)^{n-1} \right) - \left( \frac{d^{n-1}}{dx^{n-1}} ((-1)^2-1)^{n-1} \right) \right]$$

$$P_n(x) = \frac{1}{2^n . n!} \left\{ \cancelto{0}{\left( \frac{d^{n-1}}{dx^{n-1}} (0)^{n-1} \right)} - \cancelto{0}{\left( \frac{d^{n-1}}{dx^{n-1}} (0)^{n-1} \right)} \right]$$

$\Rightarrow P_n(x) = 0$

when $n = 0$

since $P_0(x) = 1$

Integrating both sides yields

$$\int_{-1}^{1} P_0(x) dx = \int_{-1}^{1} 1 \cdot dx$$

$$\int_{-1}^{1} P_0(x) dx = \int_{-1}^{1} dx$$

$$\therefore P_0(x) = \left[ x \right]_{-1}^{1}$$

$$P_0(x) = 1 - (-1) = 2$$

$\Rightarrow P_0(x) = 2$ Solved

## QUESTION 5C

Given that $P_3(x) = \frac{1}{2}(5x^3-3x)$ and

$P_4(x) = \frac{1}{8}(35x^4-3x^2+3)$ use the

recurrence relation to find $P_5(x)$.

⊛ Here, we use the recurrence relation that says

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

we take $n = 4$ into the ff relation

$$(4+1)P_{4+1}(x) = (2(4)+1)xP_4(x) - 4P_{4-1}(x)$$

$$5P_5(x) = 9xP_4(x) - 4P_3(x)$$

but $P_3(x)$ and $P_4(x)$ were given in the ① question, so we substitute

$$5P_5(x) = 9x\left[\frac{1}{8}(35x^4-3x^2+3)\right] - 4\left[\frac{1}{2}(5x^3-3x)\right]$$

$$5P_5(x) = \frac{9x}{8}(35x^4-3x^2+3) - 2(5x^3-3x)$$

$$5P_5(x) = \frac{315x^5-270x^3+27x}{8} - \frac{10x^3-6x}{1}$$

Lcm

$$5P_5(x) = \frac{315x^5-270x^3+27x-80x^3-48x}{8}$$

$$5P_5(x) = \frac{315x^5 - 350x^3 + 75x}{8}$$

Or

$$5P_5(x) = \frac{1}{8}(315x^5 - 350x^3 + 75x)$$

Divide both sides by 5

$$\frac{5P_5(x)}{5} = \left[\frac{1}{8}\left(\frac{315x^5}{5} - \frac{350x^3}{5} + \frac{75x}{5}\right)\right]$$

$$\Rightarrow \cancel{5P_5} \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$ Solved

SUNNYDAN WISHED YOU WELL IN YOUR EXAMS

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