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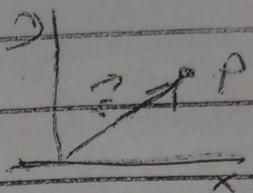
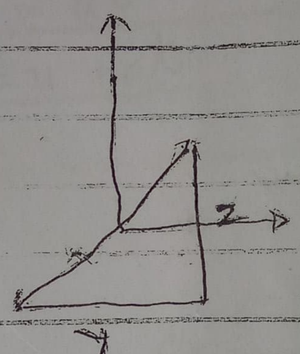
2-D motion

This is motion in a plane often defined by a locus consisting of two points



$\vec{r} = (x, y)$:= position vector of the particle.

3-D motion



$\vec{r}$:= position vector.

For 1-D

$$f = ma$$

$$v = \frac{dx}{dt} \text{ for small displacements}$$

$$\int_0^x dx = \int_0^t v\,dt$$

$$x = \int_0^t (u + at)\,dt$$

$$x = ut + \frac{1}{2}at^2$$

$$a = \frac{dv}{dt}$$

$$\int_u^v dv = \int_0^t a\,dt \quad ; \text{ a assumed constant}$$

From where we obtain

$$v - u = at$$

$$v = u + at$$

$$a = \frac{dv}{dt} \qquad v = \frac{dx}{dt}$$

$$\frac{v}{a} = \frac{dx}{dv}$$

or

$$a\,dx = v\,dt$$

$$\int_u^v v\,dv = \int_0^x a\,dx \quad ; \text{ a assumed constant}$$

$$\frac{1}{2}(v^2 - u^2) = ax$$

or

$$v^2 = u^2 + 2ax.$$

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SIMPLE HARMONIC MOTION

Oscillatory or vibrational motion constitutes one of the most important motions encountered in nature. A particle is said to be oscillating when it is moving periodically about an equilibrium position eg pendulum, atoms in molecules etc.

Simple harmonic motion, SHM is the most important of all oscillatory motion due to the simplicity of its mathematical formulation and its accurate description of many oscillations found in nature.

KINEMATICS OF SIMPLE HARMONIC MOTION

A particle moving along x-axis say is said to be performing SHM if its displacement x relative to the origin of the coordinate system is given as a function of time by

$$x = A \sin(\omega t + \alpha) \qquad (7)$$

where

$\omega t + \alpha$:= phase

α := initial phase ie the value for $t = 0$

(7) can be expressed as a cosine function (taking into acount the $\pi/2$ phase difference). Since these two functions (sine or cosine)

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varie between ± 1, then the displacement of the particle given by (7) varies between $\pm A$ where A is the maximum displacement of the particle from the origin. Since the sine function repeats itself after a time interval of $2\pi/\omega = T$, the velocity of the particle is thus

$$v = \frac{dx}{dt} = \omega A \cos(\omega t + \alpha) \quad (8)$$

$$a = \frac{dv}{dt} = -\omega^2 A \sin(\omega t + \alpha)$$
$$= -\omega^2 x \quad (9)$$

Recall that

$$\omega = \frac{2\pi}{T} \quad \text{and} \quad \frac{1}{T} = \nu$$

$$\therefore \omega = 2\pi\nu$$

$\Rightarrow$ to arrive at (9), eqn (7) had been used.

Remark

a is always proportional and opposite to the displacement.

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FORCE AND ENERGY IN SHM

The force, F acting on a mass, m performing SHM can be calculated from (9)

$$F = ma$$

using (9) we have

$$F = -m\omega^2 x$$
$$= -Kx \quad (10)$$

where

$K = m\omega^2$ = spring constant

$$\therefore \omega = \sqrt{\frac{K}{m}} \quad (11)$$

Remark

1. Eqn (10) reveals that the force is always proportional to the displacement and opposed to it i.e it is always pointing toward the origin
2. The origin here refers to the equilibrium since at the origin F = 0 whenever x = 0. F may also be thought of as attractive with the centre of attraction is the origin.
3. $\omega = \frac{2\pi}{T}$ or $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}}$ (12)

$$\nu = \frac{1}{2\pi}\sqrt{K/m} \quad (13)$$

(12) and (13) are the period and frequency of a Setting in and the elastic Constant of

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THE KINETIC ENERGY

$$T = \frac{1}{2} m v^2$$
$$= \frac{1}{2} m \omega^2 A^2 \cos^2(\omega t + \alpha)$$

or ~~putting~~

$$\cos^2(\omega t + \alpha) = 1 - \sin^2(\omega t + \alpha)$$

$$T = \frac{1}{2} m\omega^2 A^2 \left[1 - \sin^2(\omega t + \alpha) \right]$$
$$= \frac{1}{2} m\omega^2 A^2 - \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \alpha)$$

~~Recall~~

Recall that

$$x = A \sin(\omega t + \alpha)$$
$$x^2 = A^2 \sin^2(\omega t + \alpha)$$

$$\Rightarrow T = \frac{1}{2} m\omega^2 A^2 - \frac{1}{2} m\omega^2 x^2$$
$$= \frac{1}{2} m\omega^2 (A^2 - x^2) \quad (14)$$

(14) is maximum at the centre $(x = 0)$ and zero at the extremes of oscillation $(x = \pm A)$

THE POTENTIAL ENERGY

We define the potential energy E_p as

$$F = -\frac{dE_p}{dx}$$

$$-Kx = -\frac{dE_p}{dx}$$

$$\therefore \int_0^{E_p} dE_p = \int_0^x Kx\,dx$$

$$E_p = \tfrac{1}{2}Kx^2 = \tfrac{1}{2}m\omega^2x^2$$

$$= \tfrac{1}{2}m\omega^2A^2\sin^2(\omega t + \alpha)$$

Remark

1. E_p is minimum at the centre (when $x=0$) and increases as the particle approaches either extremes of the oscillation ($x = \pm A$)

2. $E = T + E_p$

$$= \tfrac{1}{2}m\omega^2(A^2 - x^2) + \tfrac{1}{2}m\omega^2x^2$$

$$= \tfrac{1}{2}m\omega^2(A^2 - x^2 + x^2)$$

$$= \tfrac{1}{2}m\omega^2A^2$$

$$= \tfrac{1}{2}KA^2$$ = constant b/c the force is conservative

3. We may therefore conclude that there is a continuous exchange of kinetic and potential energies during an oscillation

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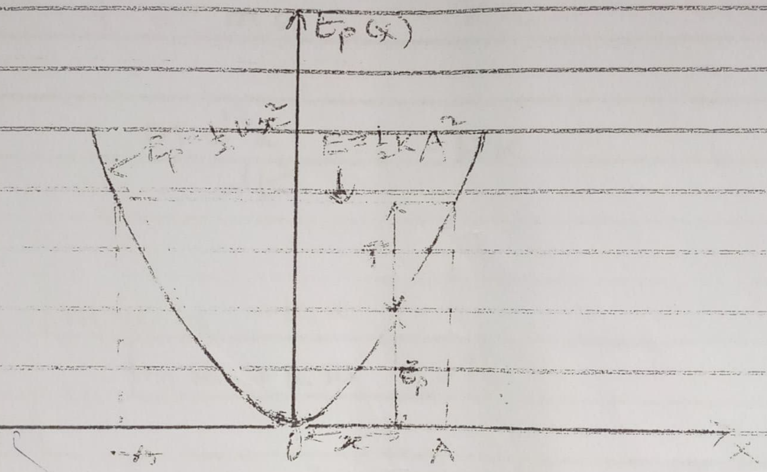
ie away from the equilibrium point, the potential energy increases at the expense of the kinetic energy and vice-versa.

(4) The equation of a potential energy is that which satisfies a parabolic curve ie



For a given total energy, E represented by the horizontal line, the limits of oscillation are determined by its intersections with the potential energy curve. Since the parabola E_p is symmetric, the limits of oscillations are at equal distances $\pm A$ from O. At any point x, the kinetic energy, T is given by the distance between $E_p(x)$ and the line E

$\frac{k}{m} \sim \sqrt{\frac{k}{m}}$

$w = 2\pi f$

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DYNAMICS OF SHM

From the foregoing, one can say that given an attractive force proportional to the displacement (ie $F = -Kx$) the resulting motion is SHM ie

$$F = ma = -Kx$$

$$m\frac{d^2x}{dt^2} = -Kx$$

$$m\ddot{x} = -Kx$$

$$m\ddot{x} + Kx = 0$$

or

$$\ddot{x} + \omega^2 x = 0 \qquad \omega^2 = \frac{K}{m} \quad (17)$$

(17) has a solution that is satisfied by the harmonic functions ie sine/cosine functions. By substituting $A\sin(\omega t + \alpha)$ for x it can be found that this expression for x which corresponds to SHM satisfies (17). Thus

$$x = A\sin(\omega t + \alpha)$$

is the general solution

WARNING!

This situation applies to numerous physical situation. Whenever it is found. it indicates that the corresponding phenomenon is oscillatory according to (17) whether it is

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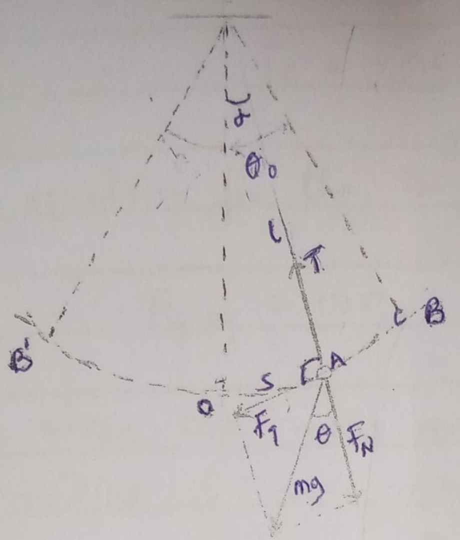
describing a linear or angular displacement of a particle of current in an electric cct, the ion concentration in plasma, the temperature of a body etc.

Examples of SHM

EXAMPLES OF SHM



If the mass in the diagram above is pulled aside to a position B so that the string makes an angle θ_0 with the vertical OC, and then released, the pendulum will oscillate between B and the symmetric position B'. To determine the nature of oscillations we must write the equations of motion of the particle. The particle moves in an arc of a circle of radius $l = OA$. The forces acting on the mass are

(i) its weight mg

(ii) Tension, T along the string

The tangential component of the resultant force

F_T can be obtained from the figure

$$F_T = -mg\sin\theta$$

-ve bc it is opposed to the displacement s=lθ

The equation for the tangential motion is

$$F_T = ma_T$$

and since the mass moves along a circle of radius l, we may write

$$a_T = l\frac{d^2\theta}{dt^2} \quad \text{where } R = l$$

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \qquad (18)$$

(18) is not an equation of SHM due to the presence of $\sin\theta$. However, if θ is small which is true provided the amplitude of oscillation is small, then

$$\sin\theta \approx \theta$$

and (18) becomes

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad (19)$$

which is a proper equation for SHM

3

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$\ddot{\theta} + \omega^2 \theta = 0$. $\omega^2 = g/l$.

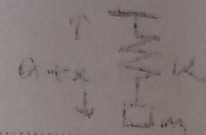
In general

$$\theta = \theta_0 \sin(\omega t + \alpha)$$

$$T = 2\pi/\omega = 2\pi\sqrt{\frac{l}{g}}$$

Remarks

1. T is independent of the mass of the pendulum.
2. For large amplitudes $\sin\theta \approx \theta$ is not valid
3. T depends on θ_0 where θ_0 is the amplitude.



k is the restoring force per displacement

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Question No: Example for a spring.

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Before loading : $l_{spring} = a$
After " : " $= a + x$

E_p of the spring $= \frac{1}{2}Kx^2$
" " " mass $= mg(x+a)$

$V_{total} = \frac{1}{2}Kx^2 - mg(x+a)$

At equilibrium

$$\frac{dV}{dx} = 0$$

$\Rightarrow \quad Kx - mg = 0$

Equilibrium distance, x_{eq} is

$$x_{eq} = \frac{mg}{K}$$

Motion of the mass about equilibrium

$$m\ddot{x} = mg - F$$
$$= mg - kq$$

$q = x_{eq} + x$
$= \frac{mg}{K} + x$

$\therefore \quad m\ddot{x} = mg - k\left(\frac{mg}{k} + x\right)$

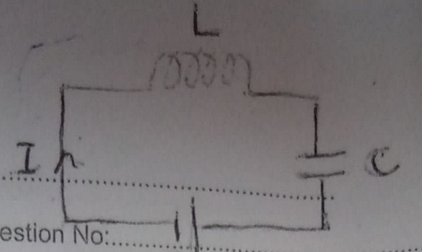
or

$$m\ddot{x} + Kx = 0$$

or $\quad \omega^2 = \frac{K}{m}$

$\ddot{x} + \omega^2 x = 0.$

$\ddot{x} + \omega^2 x = 0.$



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$$V_L = L\frac{d^2I}{dt^2}$$

$$V_c = \frac{I}{C}$$

At resonance

$$V_L = -V_c$$

$$\therefore L\frac{d^2I}{dt^2} + \frac{I}{C} = 0$$

$$\frac{d^2I}{dt^2} + \frac{I}{LC} = 0$$

or

$$\omega = \sqrt{\frac{1}{LC}}$$

$$T = 2\pi\sqrt{LC}$$

Assignment

1. In a certain automobile, when five passengers get in, their total mass being 360 kg, the spring of the suspension are compressed a distance 4 cm. The total mass including the passengers supported by the suspension is 900kg. Assuming that the displacement of the springs is directly proportional to the compressive force in them, calculate the period of oscillation of the loaded car on its suspension, neglecting any effect of damping (which of course is very heavy in a well designed suspension system) $g = 10\,N\,Kg^{-1}$

2. A typical mountaineer's rope about 35m long will give a distance of about 1.6m under the weight of a climber hanging freely on the end. Assuming the climber's mass is 80kg, estimate the period of vertical oscillation of the climber when dangling freely on the end of the rope How does this compare with the period of swing of the climber "pendulum"

3.

DAMPED SIMPLE HARMONIC MOTION

In ~~the~~ last discussion (i.e. ideal SHM) ~~the~~ basic assumptions is that the ~~the~~ total energy of the system remains constant (i.e. there is no energy loss due to frictions or any other resistances to motion) and the displacement followed a sine curve apparently for a infinite long time.

In any real-life mechanical oscillatory system performing free oscillations, there are resistances to motions which cause a loss of energy and a decay of the amplitude of oscillations e.g. the amplitude of a freely swinging pendulum will always decay with time as energy is lost. Oscillations decaying in this way are called DAMPED " and the term DAMPED of damping is generally used to signify how rapidly the amplitude decays. The presence of resistance to motion means that another force is active, which is taken as being proportional ~~to the~~ velocity. Like the stiffness force, it always acts in a directn opposite to that of the acceleration term. The eqn. of motion thus becomes

$$m\ddot{x} = -Kx - r\dot{x}$$

where r is the constant of proportionality and has dimensions of force per unit of velocity. The presence of such term will always result in energy loss.

PROBLEM. To find the displacement x from

$$m\ddot{x} + Kx + r\dot{x} = 0 \quad (7) \rightarrow \text{second order diff Equation}$$

m, r, K being constants

When these coefficients are constant a solution of the form

$$x = Ce^{\alpha t}$$

can always be found. Since an exponential term is always non dimensional, C has the dimensions of x (length, say) and α has the dimensions of inverse time, T^{-1}. We shall see that there are three possible forms of this solution, each describing a different behaviour of the displacement x with time. In two of these solutions C appears explicitly as a constant of length, but in the third case, it takes the form

$$C = A + Bt$$

Remark.

The number of constants allowed in the general solution of a differential eqn is always equal to the order (i.e the highest differential coefficient) of the eqn. Th two values A & B are allowed because (9) is 2nd order. The values of the constants are adjusted to satisfy the initial conditions.

$m\ddot{x} + r\dot{x} + s x$

C as a constant length then

Taking

$$\dot{x} = \alpha Ce^{\alpha t}$$

$$\ddot{x} = \alpha^2 Ce^{\alpha t}$$

so that (9) becomes

$$Ce^{\alpha t}(m\alpha^2 + r\alpha + s) = 0$$

so that either

$$x = Ce^{\alpha t} = 0 \quad \text{(which is trivial)}$$

or

$$ax^2 + bx + c = 0$$

$$m\alpha^2 + r\alpha + K = 0$$

Solving the quadratic eqn in α yields

$$\alpha = -\frac{r}{2m} \pm \sqrt{\frac{r^2}{4m^2} - \frac{K}{m}}$$

Remark

1. $(r/2m)$; $\left(\frac{K}{m}\right)^{1/2}$ & α have the dimensions of T^{-1}

2. $$x = Ce^{-\frac{rt}{2m} \pm \left(r^2/4m^2 - K/m\right)^{1/2} t} \quad (9a)$$

where $\left(\frac{r^2}{4m^2} - \frac{K}{m}\right)$ can be +ve, zero or -ve depending on the relative magnitude of the two terms inside it.

3. Each of these conditions gives one of the three possible solutions referred to earlier and each soln describes a particular kind of behaviour

Damping is an influence within or upon an oscillatory system that has the effect of reducing, restricting or preventing it oscillations. It is produced by processes that dissipate the energy stored in the oscillation.

Heavy damping (10)

$r^2/4m^2 > k/m$ | Heavy Damping

Case I If the Bracket +ve ie

Here the damping resistance term $r^2/4m^2$ dominates the stiffness term k/m and heavy damping results in a dead beat system.

Writing

$$\frac{r}{2m} = p$$

and

$$\left(\frac{r^2}{4m^2} - \frac{k}{m}\right)^{1/2} = q$$

we have that (9a) can be replaced by

$$x = e^{-pt}\left(C_1 e^{qt} + C_2 e^{-qt}\right) \quad (9b)$$

We recall that the exponential is in two parts ie both possible solutions ie

where the C_i's are arbitrary in value but have the same dimensions as C (two separate values of C are allowed bc the d.e (9) is second order).

Now if

$$F = C_1 + C_2$$
$$G = C_1 - C_2$$

then

$$F + G = 2C_1$$

or

$$C_1 = \frac{F+G}{2}$$

Similarly

$$F - G = 2C_2$$

$$\frac{F-G}{2} = C_2$$

Substituting in (9b) we have

$$x = e^{-pt}\left\{\frac{F+G}{2}\left(e^{qt}\right) + \frac{F-G}{2}\left(e^{-qt}\right)\right\}$$

$$e^{-pt}\left\{\frac{Fe^{qt}}{2} + \frac{G}{2}e^{qt} + \frac{F}{2}e^{-qt} - \frac{Ge^{-qt}}{2}\right\}.$$

collecting like terms we have

$$x = e^{-pt}\left[\frac{F}{2}\left(e^{qt}+e^{-qt}\right) + \frac{G}{2}\left(e^{qt}-e^{-qt}\right)\right] \quad (**)$$

or

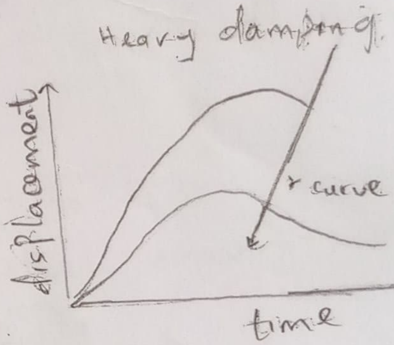
$$x = e^{-pt}\left\{F\cosh qt + G\sinh qt\right\}$$

Assignment III - Prove eqn (**)

This represents non-oscillatory behaviour, but the actual displacement will depend upon the initial (or boundary conditions, ie the value of x at time $t=0$. If $x=0$ at $t=0$ then $F=0$ and

$$x = Ge^{-rt/2m}\sinh\left(\frac{r^2}{4m^2}-\frac{k}{m}\right)^{1/2}t$$

Remark: Physical Consequence



* A heavily damped system when suddenly disturbed from equilibrium by a sudden impulse (ie given a velocity at $t=0$) will return to zero displacement quite slowly without oscillating about its equilibrium position.

CASE II: CRITICAL DAMPING ($r^2/4m^2 = k/m$)

In this case the bracket is zero and the balance in the two terms results in a critically damped system. Noting the notations as in case I we see that $q = 0$ and so

$$x = Ce^{-pt}$$

This is in fact the limiting case of the behaviour of case I as q changes from +ve to negative. In this case the quadratic eqn in x has equal roots, which, in a d.e soln demands that C be written as

$$C = A + Bt$$

where A is a constant length and B is given velocity which depends on the boundary condns.

Thus we have

$$x = (A + Bt)e^{-rt/2m} = (A + Bt)e^{-pt}$$

APPLICATION TO A BALLISTIC GALVANOMETER

Critical damping plays a key role of practical importance in recording instruments such as ballistic galvanometers which experience sudden impulses and are required to return to zero displacement at t = 0. in the minimum time.

Suppose such a galvanometer has zero displacement at $t = 0$ and receives a quantity of electric charge which gives its light spot an initial velocity v over a linear scale.

...hen $x = 0$ (so that $A = 0$) since A has the dimensions of lengths as x.

and

$$\dot{x} = v \quad \text{at } t = 0.$$

However

$$\dot{x} = \frac{d}{dt}\left(B t e^{-pt}\right)$$ ← use chain rule (differentiate of a product).

(*) $$= B\{(-pt)e^{-pt} + e^{-pt}\} = B \quad \text{at } t = 0$$

So that

$$B = v$$ and

$$x = vte^{-pt}$$

The maximum displacement, x occurs when the instrument light spot comes to rest before returning to zero displacement. At maximum displacement

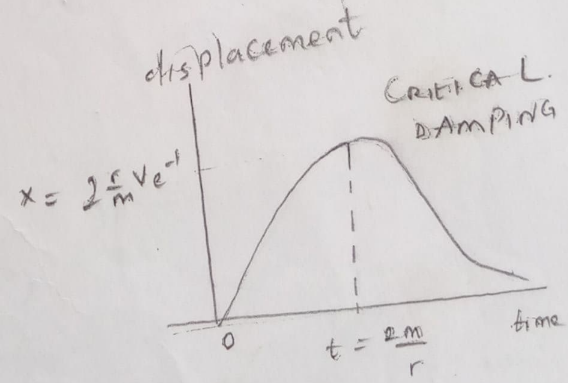
$$\dot{x} = ve^{-pt}(1 - pt) = 0 \quad \text{. see (*) above}$$

this gives

$$1 - pt = 0$$

or

$$t = \frac{1}{p}$$



At this time,

$$x = vte^{-pt} = \frac{v}{p}e^{-1}$$

$$= 0.368\,\frac{v}{p} = 0.368\,\frac{2mv}{r}$$

CASE III - DAMPED SHM ie $r^2/4m^2 < k/m$

This is the situation in which the damping is light, and bracket is negative. The this is of high practical importance.

The expression $(r^2/4m^2 - k/m)^{1/2}$ is an imaginary quantity, the square root of a negative number which can be rewritten

$$\pm\left(\frac{r^2}{4m^2} - \frac{k}{m}\right)^{1/2} = \pm\sqrt{-1}\left(\frac{k}{m} - \frac{r^2}{4m^2}\right)^{1/2}$$

$$= \pm i\left(\frac{k}{m} - \frac{r^2}{4m^2}\right)^{1/2} \quad (\text{where } i = \sqrt{-1})$$

so that

$$x = C e^{-rt/2m} e^{\pm(k/m - r^2/4m^2)^{1/2} t}$$

Remark

1. The bracket has the dimensions of T^{-1} ie that of frequency and hence can be written

$$\left(\frac{k}{m} - r^2/4m^2\right)^{1/2} = w'$$

so that $e^{\pm(k/m - r^2/4m^2)^{1/2} t} = e^{\pm iw't}$

$$= \cos w't + i\sin w't.$$

2. This shows that the behaviour of the displacement x is oscillatory with a new frequency

$$w' < w = (k/m)^{1/2}$$

the frequency of an ideal SHM.

3. To compare the behaviour of the damped oscillator with the ideal case we should take

to express the soln. in a form similar to

$$x = A \sin(\omega' t + \phi)$$

as in ideal case, where ω has been replaced by ω'. We can do this by writing

$$x = C e^{-rt/2m} e^{\pm i\omega' t}$$

$$= e^{-rt/2m}\left(C_1 e^{i\omega' t} + C_2 e^{-i\omega' t}\right)$$

If we now choose

$$C_1 = \frac{A}{2i} e^{i\phi}$$

and

$$C_2 = -\frac{A}{2i} e^{-i\phi}$$

where A, ϕ (and $e^{i\phi}$) are constants which depend on the motion at $t = 0$, we find after substitution

$$x = A e^{-rt/2m} \left[\frac{e^{i(\omega' t + \phi)} - e^{-i(\omega' t + \phi)}}{2i}\right] \quad (B)$$

$$= A e^{-rt/2m} \sin(\omega' t + \phi)$$

Assignment IV: Show the detailed steps that led to eqn (B)

This procedure is equivalent to imposing the boundary conditions

$$x = A \sin \phi \quad \text{at } t = 0$$

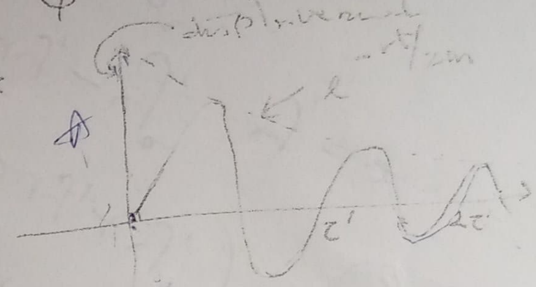
upon the solution for x. The displacement therefore varies sinusoidally with time as in the case of SHM but now has a new frequency

$$\omega' = \left(\frac{k}{m} - \frac{r^2}{4m^2}\right)^{1/2}.$$

and its amplitude A is modified by the

exponential term $e^{-rt/2m}$, a term which decays with time.

4. If $x = 0$ at $t = 0$ then $\phi = 0$



The figure shows ~~the behaviour of x~~ with time, its oscillations gradually decaying with the envelope of maximum amplitudes following the dotted curve $e^{-rt/2m}$. The constant A is the value to which the amplitude would have risen at the first maximum if no ~~dam~~ damping were present

5. The presence of the force term $r\dot{x}$ in the eqn of motion therefore introduces a loss of oscillatory energy which causes the amplitude of oscillation to decay with time as $e^{-rt/2m}$.

RIGID BODY DYNAMICS

This deals essentially with Rotational kinetic energy where every particle within the object is moving as different from translation K.E, that an object possesses by virtue of its translational motion.

$$(T)_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 r^2 w^2 = \frac{1}{2} m_1 r^2 \dot{\theta}^2 \quad (A.1)$$

where we have used

$$v = r w, \quad w = \dot{\theta} \quad (A.2)$$

where θ is the angle subtended by the rotating particle

$$(T)_s = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \left(\frac{1}{2} m_i r_i^2\right) \dot{\theta}_i^2 \quad A.3$$

For rigid systems, $\dot{\theta}_1 = \dot{\theta}_2 = \dots$ because d_{ij} = constant

We may thus write

$\Sigma m_i r_i^2 \to$ suming discrete cases.

$\int r^2 dm \to$ suming continous cases.

$$(T)_s = \left(\sum_{i=1}^{N} m_i r_i^2\right) \frac{w^2}{2}$$

$$= \frac{1}{2} I w^2 \quad (A.4)$$

where

$$I = \sum m_i r_i^2 := \text{moment of inertia.} \quad (A.5)$$

Since one is not found in any way in the above discussion concerning the quantity of mass m_i, it means that m_i could be varied as we please ie it could be made infinitesimal and we can write

$$I = \int r^2 dm$$

CONTACT WITH CIRCULAR MOTION

Call

$$\sum_{i=1}^{n} m_i = \int dm = M.$$

Call

$$\frac{I}{M} = \frac{\int r^2 dm}{\int dm} = \frac{\sum m_i r_i^2}{\sum m_i} = K^2 \quad \text{~~(A-6)~~}$$

Then

$$I = MK^2 \quad \text{- - - - - (A-6)}$$

and

$$T = \frac{1}{2} I w^2 = \frac{1}{2} M K^2 w^2 = \frac{1}{2} M v^2 \text{ . . . (A-7)}$$

where T is ~~the rotational~~ kinetic energy. K has the dimensions of length ie K = r ie we can relate Kw with rw = v of circular motion. We have thus reduced this N-particle rotations about O to that of a particle of mass M placed a distance K from O - which essentially constitutes circular motion of point mass having a circle radius a K

K := radius gyration which ~~is the distance from the point~~ of rotation at which the system can be considered placed for ~~the purpose~~ of dynamical analysis.

COMPARATOR TABLE. Scale parameter, r

Linear motions	Angular motions	Connection
x	θ	$x = r\theta$
v	ω	$v = r\omega$
a	α	$a = r\alpha$
M	I	$I = Mr^2$

Derived quantities

$T = \frac{1}{2} Mv^2$	$\frac{1}{2} I\omega^2$
$F = ma$	$\Gamma = I\alpha$

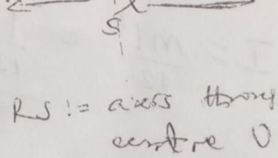
↳ Torque (moment of a force)

METHODS FOR CALCULATING, I

(I) $I = \int r^2 \, dm$

Example.

I of uniform rod about axis through middle



RS := axis through centre O

δx → a small element

We define the mass per unit length $= \sigma$

$\therefore \frac{M}{l} = \sigma$

$M = l\sigma$

$dm = \sigma \, dx$

$$\int_{-l/2}^{l/2} x^2 \sigma \, dx = 2\left(\sigma \frac{x^3}{3}\right)\Big|_0^{l/2} = 2\left(\frac{\sigma (l/2)^3}{3}\right) = \frac{\sigma l^3}{4 \cdot 3} = \frac{\sigma l^3}{12}$$

About the axis through one end

$$I = \int_0^l x^2 \sigma \, dx = \frac{\sigma l^3}{3}$$

Example: Uniform circular disc

Let $q = \frac{M}{A} = \frac{M}{\pi R^2}$

$\Rightarrow qA = M.$

$A = \pi R^2 \quad \pi r^2$

$dA = 2\pi r \, dr$

$dm = d(q \pi r^2)$

$= q \, 2\pi r \, dr$

$I = q \int r^2 \, dA$

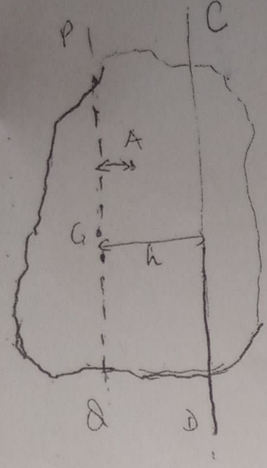
$= \int_0^R r^2 \cdot 2\pi r \, dr$

$I = 2\pi q \int_0^R r^3 \, dr$

$I = 2\pi q \frac{R^4}{4}$

$I = \frac{q \pi R^4}{2} = \frac{MR^2}{2}$

PARALLEL AXIS THEOREM



I := moment of inertia of a body about C

I_G := " " " about a parallel axis PQ thro' the centre of gravity,

h := distance between the two axes that are parallel to one another.

m := mass of the particle A.

x := distance of A from PQ.

I_p about CD $= m(h-x)^2$

$\therefore I_p = \sum m(h-x)^2 = \sum mh^2 + \sum mx^2 - \sum 2mhx$

$\sum mh^2 = h^2 \times \sum m = Mh^2$

M := total mass of the object

$\sum mx^2 := I_G$ = moment of inertia through the centre of gravity

Also $\sum 2mhx = 2h \sum mx = 0$

Reason being that the sum of the moments about G is zero.

$\therefore I = I_G + I_h$

where $I_h = Mh^2$

PERPENDICULAR AXIS THEOREM

A general rotation in three dimensions ~~these are the motions~~ about the three orthogonal axes mutually at right angles to themselves. Around ~~these~~ axes is

$$I = \sum_{i=1} m_i r_i^2 = \sum m_i (x_i^2 + y_i^2 + z_i^2)$$

$$= \sum m_i x_i^2 + \sum m_i y_i^2 + \sum m_i z_i^2$$

$$= I_x + I_y + I_z$$

ROUTH'S RULE {NOT APPLICABLE TO HOLLOW BODIES}

The moment of inertia I is given by

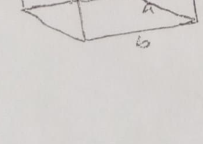
$$I = \frac{\text{Sum of the squares of } \perp\text{ar semi-axis}}{3, 4, \text{ or } 5}$$

3 if semi-axis terminate on a plane surface

4 if " " " " cylindrical surface

5 " " " " " doubly curved surface

Example 1.



$$I = \frac{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2}{3} \times M$$

$$= M \frac{a^2 + b^2 + c^2}{12}$$

2. for a doubly curved surface of solid sphere

$$I = M \left\{ \frac{R^2 + R^2 + R^2}{5} \right\}$$

$$= \frac{3}{5} MR^2$$

LEE'S RULE:

$$\bar{I} = m\left\{\frac{a^2}{3+n_1} + \frac{b^2}{3+n_2}\right\}$$

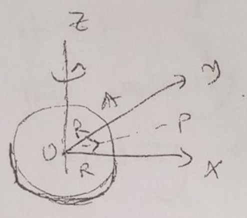
where a, b are semi-lar axis

$n_1 = n_2 = 0$ if any axis terminate in a plane

$n_1 = n_2 = 1$ if one or both terminate on a cylindrical surface

$n_1 = n_2 = 2$ if any of them terminate on a doubly curved surface.

APPLICATIONS



CIRCULAR DISC

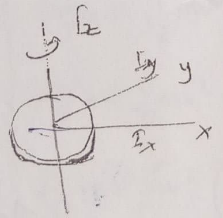
Routh Rule produces

$$I = M\left(\frac{R^2}{4} + \frac{R^2}{4}\right) = \frac{1}{2} MR^2$$

LEE'S RULE

$$I = m\left(\frac{R^2}{3+1} + \frac{R^2}{3+1}\right) = \frac{1}{2} mR^2.$$

DISC ABOUT DIAMETER



$$I_z = I_x + I_y = 2I$$

$$= \frac{1}{2} mR^2$$

$$\therefore I = \frac{1}{4} MR^2$$

Assignment 7 8

1. A solid sphere of mass M and radius R is rolling down an inclined plane without slipping. Find the speed of the centre of mass when the ~~cylinder~~ sphere reaches the bottom.

2. A sphere and a cylinder having same mass and radius starts from rest and roll down the same incline. Which one reaches the bottom first.

3. A solid sphere of radius R and mass M starts moving in a straight line path (no rolling) with an initial velocity $\vec{v}_0$. After some time it starts rolling without skidding. Find the velocity of the centre of sphere when it is rotating down in straight line.

(1) Soln

$P.E = K.E$

$K.E = \text{Tran} + \text{Rot}$ of sphere

$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

I of sphere $= \frac{2}{5}mR^2$

$\omega = \frac{v}{R}$

$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\frac{v^2}{R^2}$

$mgh = \frac{1}{2}mv^2 + \frac{1}{5}mv^2$

$gh = v^2\left(\frac{1}{2} + \frac{1}{5}\right)$

$2(5+2)$

$P.E = K.E$

$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

$I = \frac{1}{2}mR^2 \quad \omega = \frac{v}{R}$

$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\frac{v^2}{R^2}$

$mgh = \frac{1}{2}mv^2 + \frac{1}{4}mv^2$

$mgh = mv^2\left(\frac{1}{2} + \frac{1}{4}\right)$

$gh = v^2\left(\frac{2+1}{4}\right)$

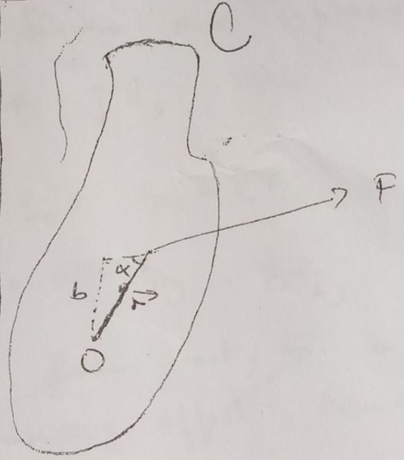
$gh = v^2\left(\frac{3}{\ }\right.$

$K.E =$

FORCES ACTING ON A RIGID BODY

When a force acts on an extended body, the body does not merely move in the direction of the force but usually changes its orientation by turning. Consider a force F acting on a body C. Suppose the effect of the body is to rotate the body around O

(2) DEFN:- RIGID BODY
Rigid body is a system of particles in which the distances $\vec{r}_{ij}$ are fixed and cannot vary with time ie if the distances b/w the particles (atoms) making it up maintains a constant distance with time.



We define the torque, τ as

Torque is the cross product of force and distance ⊥ to the system.

$$\tau = \text{force} \times \text{perpendicular distance.}$$

b := ⊥ar distance to the force, F.

$\vec{r}$:= position vector or radius vector.

from the fig above

$$b = r \sin\alpha$$

$$\therefore \tau = bF = r \sin\alpha\, F$$

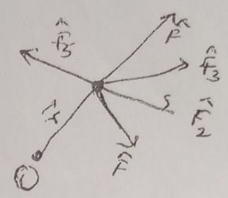
In vector notation, this becomes

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}} \quad (13)$$

Remark

① $\vec{\tau}$ is ⊥ar ~~to the~~ plane formed by $\vec{r} \wedge \vec{F}$

② If all the forces are concurrent i.e. they are all applied ~~at the~~ same point, then the resultant R is given as



$$\hat{R} = \hat{F}_1 + \hat{F}_2 + \cdots + \hat{F}_N$$

$$= \sum F_i$$

If all the forces are coplanar in the XY plane i.e.

$$\vec{R} = \hat{i} R_x + \hat{j} R_y$$

where $R_x = F_{1x} + F_{2x} + \cdots F_{Nx} = \sum F_{ix}$

$R_y = F_{1y} + \cdots + F_{Ny} = \sum F_{iy}$

then ~~and if such forces act in a such a way as to produce a turning effect then~~

$$|\hat{R}| = \left(R_x^2 + R_y^2\right)^{1/2}$$

$$\tan\alpha = \frac{R_y}{R_x}.$$

If such forces act in a such way as to produce a turning effect then

$$\hat{\tau} = \hat{r} \wedge \hat{R}$$

$$= \hat{r} \times \sum_i F_i$$

$$= \hat{r} \wedge \left[\hat{F}_1 + \hat{F}_2 + \cdots + \hat{F}_N\right]$$

$$= \hat{\tau}_1 + \hat{\tau}_2 + \cdots + \hat{\tau}_N$$

$$= \sum_i \tau_i \quad (14)$$

③ (14) shows that a system of concurrent forces can be replaced by a single force, its resultant which is completely equivalent to the system in so far as the translational and the rotational effects are concerned

COMPOSITION OF FORCES APPLIED TO A RIGID BODY

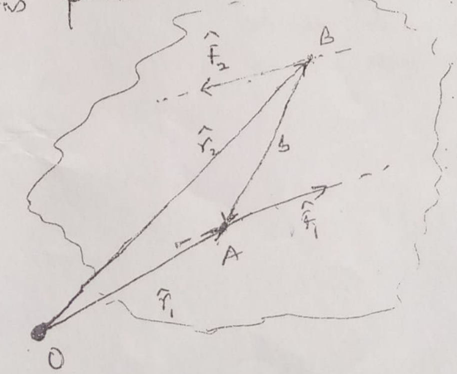
The above statement is not true in general ie a system of forces acting on rigid body cannot be reduced to a single force or resultant equal to the vector sum of the forces, eg a couple.

COUPLE :- A couple is defined as a system of two forces of equal magnitude but opposite directions acting along parallel lines.

The resultant or vector sum of the 2 forces = 0.

$$\hat{R} = \hat{f}_1 + \hat{f}_2 = 0$$

$\Rightarrow$ no translational effect is produced by the couple



Taking into account that

$f_2 = -f_1$ then

$$\hat{\tau} = \hat{\tau}_1 + \hat{\tau}_2$$

$$= \hat{r}_1 \times \hat{f}_1 + \hat{r}_2 \times \hat{f}_2$$

$$= \hat{r}_1 \times \hat{f}_1 - (\hat{r}_2 \wedge \hat{f}_2)$$

$$= (\hat{r}_1 - \hat{r}_2)\hat{f}_1$$

$$= \hat{b} \times \hat{f}_1$$

(15)

$\hat{b} = \hat{r}_1 - \hat{r}_2$ and is referred to as LEVER ARM of the couple.

$\Rightarrow \hat{\tau} \neq 0$ even though $\hat{R} = 0$; hence produces a rotational effect.

REMARK

1. $\hat{b}$ is independent of O ie the torque of a couple is independent of the origin about which the torque is computed. Obviously no single force can satisfy all these conditions

(2) In general, a system of forces acting on a rigid body can be reduced to a force and a couple. The force is chosen equal to $\hat{R}$ for translational equivalence and is applied at the point about which the torques were evaluated; so that $\hat{\tau} = 0$. The couple with a torque equal to $\hat{\tau}$ is then chosen for rotational equivalence

COMPOSITION OF PARALLEL FORCES

A system of forces parallel to a unit vector $\hat{u}$ can be represented by

$$\hat{F}_i = \hat{u} f_i$$

$\hat{f}_i$ may be positive or negative depending on whether the direction of $\hat{f}_i$ is the same as that of $\hat{u}$ or opposite. The resultant force can thus be written as

$$\boxed{\hat{R} = \hat{u} \sum \hat{f}_i} \quad (16)$$

$\hat{R} \parallel \hat{u}$ $\qquad |\hat{R}| = \sum F_i$

$\Rightarrow$

$\hat{R}$ may be considered as applied at a point ~~given by $\hat{r}_c$~~ called the centre of parallel forces and is defined by the position vector $\hat{r}_c$

$$\hat{r}_c = \frac{\sum_i \hat{r}_i F_i}{\sum_i F_i}$$

where $\hat{r}_c = r_c(x_c, y_c, z_c)$

(13)

Assignment 5

Prove that

$$\hat{r}_c = \frac{\sum \vec{r}_i f_i}{\sum f_i}$$

CENTRE OF MASS

It is a known fact that every particle within the Earth's gravitational field is acted upon by the weight, $W = mg$. The direction of this force if extended passes through the centre of the earth.

$$\hat{W} = \sum m_i \hat{g}$$

$\hat{W}$ is the resultant weight extended over all particles that comprises the body and acts through a point $\hat{r}_c$ given by

$$\hat{r}_c = \frac{\sum \hat{r}_i m_i g}{\sum m_i g} = \frac{\sum m_i \hat{r}_i}{\sum m_i} \qquad (17)$$

or

$$x_c = \frac{\sum m_i x_i}{\sum m_i} \qquad y_c = \frac{\sum m_i y_i}{\sum m_i} \qquad z_c = \frac{\sum m_i z_i}{\sum m_i} \quad \text{-- (18)}$$

where $\hat{r}_c = r_c (x_c, y_c, z_c)$.

(17) and (18) are called centre of mass of the system of particles. This concept plays an important role not only in the composition of parallel forces but also in the analysis of motion of a system of particles and in particular of rigid body

COLLISIONS

In an isolated system of particles (ie a system of particles not subject to any external actions or forces), the total momentum and energy both referred to an inertial frame of reference, remains constant. These two conservation laws are not independent because, in view of the Lorentz transformation for energy and momentum, the conservation of momentum in all inertial reference frames also require the conservation of energy ie

$$p_x' = \frac{p_x - \frac{vE}{c^2}}{(1 - v^2/c^2)^{1/2}} = \gamma\left(p_x - \frac{vE}{c^2}\right)$$

$$p_y' = p_y$$

$$p_z' = p_z$$

$$E' = \frac{E - vp_x}{(1 - v^2/c^2)^{1/2}} = \gamma(E - vp_x$$

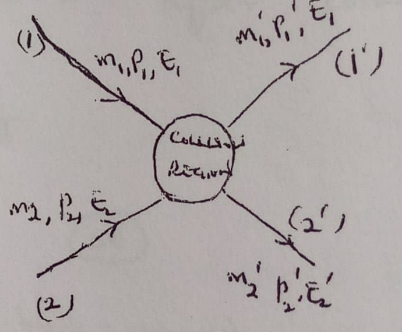
These conservations laws shall hence be applied to collision analyses.

COLLISION :- When two particles approach each other, their mutual interaction changes their motion producing an exchange of momentum & energy. This is referred to as collision.

SCATTERING :- This is a type of collision in which the initial and final particles are the same.

However in some collision experiment (ie nuclear & chemical reactions) the final particles are not identical to the initial ones.

Often times the precise motion of the particles in a collision expt is known before collision takes place since such motion depends on how the experiment has been prepared.



Example: Particle bombardment ~~in which~~ the electrostatic accelerator in which ~~the~~ atom is practically at rest in the laboratory. The particle could be an ~~atom~~ electron or proton. The motion of the two particles could then be observed at points far away from the region in which they collided ie ~~fig above~~ figure above.

Alternatively, if the forces between the particles are known, then the final states of the system could be predicted so long as we know the initial state

COMMENT : This is the usual cause and effect syndrome inferred from Newton's laws which today is not altogether true. In the first instance, it is only in few cases that we know all the forces acting on system. Secondly, the issue of predictability is a controversial one as seemingly simple systems whose forces could even be spelt out are unpredictable.

IMPORTANCE:

① Collision experiment analysis is very important because it provides valuable information about the interaction between the colliding particles.

② The momentum and the total energy are conserved in collision experiment since only internal forces are involved (isolated systems).

FORMULATION OF PROBLEM

$$\vec{p}_1 + \vec{p}_2 = p_1' + p_2' \qquad (29)$$

$\vec{p}_1$ & p_2 := momentum of each particle before collision

$\vec{p}_1'$ & $\vec{p}_2'$:= " " " after collision.

Conservation of Energy requires that

$$E_1 + E_2 = E_1' + E_2' \qquad (30)$$

where each energy is given by relativity

$$E = c\,(m_o^2 c^2 + p^2)^{1/2} \qquad (31)$$

i.e.

$$c\sqrt{m_1^2 c^2 + p_1^2} + c\sqrt{m_2^2 c^2 + p_2^2} = c\sqrt{m_1'^2 c^2 + p_1'^2} + c\sqrt{m_2'^2 c^2 + p_2'^2} \qquad (32)$$

Recalling that

$$E = T + m_o c^2$$

then

$$T_1 + m_1 c^2 + T_2 + m_2 c^2 = T_1' + m_1' c^2 + T_2' + m_2' c^2$$

If we define

Q := change in kinetic energy during the collision

$$Q = (T_1' + T_2') - (T_1 + T_2) = (m_1 + m_2 - m_1' - m_2')c^2 \quad (33)$$

= change in the rest energies during collisions.

REMARK

1) ELASTIC COLLISION

If $Q = 0$, kinetic energy is conserved in the collision and is thus elastic

2) INELASTIC COLLISION

If and when $Q < 0$, $T \downarrow$ (or $m \uparrow$) which results in inelastic collision of the first kind otherwise known as ENDOERGIC INELASTIC COLLISION.

When $Q > 0$, $T \uparrow$ (or $m \downarrow$) and we have EXOERGIC INELASTIC COLLISION

In general, for more than two particles we have

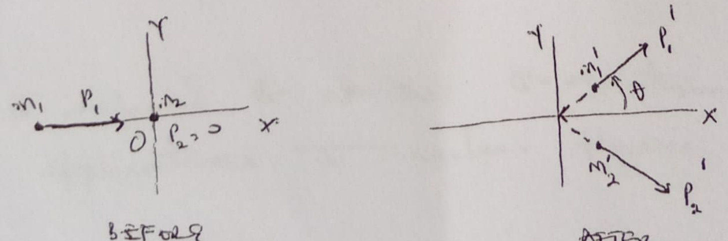
$$Q = \sum_i T_i' - \sum_i T_i = \left[\sum m_i - \sum m_i'\right] c^2 \quad (34)$$

3) When $Q < 0$, there is a minimum threshold kinetic energy of the incoming particles needed to make the collision occur or "go".

For nuclear reactions at relatively low energies, all particles may (in general) be treated non relativistically so that

$$T = \frac{1}{2m} p^2$$

Consider the following



m_1 := projectile (rest mass)

p_1 = momentum of the projectile

m_2 := rest mass of the target nucleus

$p_2 = 0$ since it is at rest in the lab frame.

After collision the particles move in different directions

Conservation of momentum gives

$$\vec{p}_1 = \vec{p}_1' + \vec{p}_2'$$

$$\vec{p}_2' = \vec{p}_1 - \vec{p}_1'$$

$$\therefore \vec{p}_2'^{\,2} = (\vec{p}_1 - \vec{p}_1')^2 = \vec{p}_1^{\,2} + \vec{p}_1'^{\,2} - 2\vec{p}_1 \cdot \vec{p}_1' \qquad (\vec{p}_1 \cdot \vec{p}_1' = p_1 p_1' \cos\theta)$$

$$= p_1^2 + p_1'^2 - 2 p_1 p_1' \cos\theta$$

Using (33) we have

$$Q = \frac{p_1'^2}{2m_1'} + \frac{p_2'^2}{2m_2'} - \frac{p_1^2}{2m_1}$$

$$= \frac{p_1'^2}{2m_1'} - \frac{p_1^2}{2m_1} + \frac{1}{2m_2'}\left(p_1^2 + p_1'^2 - 2p_1 p_1' \cos\theta\right)$$

or

$$Q = \frac{1}{2}\left(\frac{1}{m_1'} + \frac{1}{m_2'}\right)p_1'^2 + \frac{1}{2}\left(\frac{1}{m_2'} - \frac{1}{m_1}\right)p_1^2 - \frac{p_1 p_1'}{m_2'}\cos\theta$$

$$= T_1'\left(1 + \frac{m_1'}{m_2'}\right) - T_1\left(1 - \frac{m_1}{m_2'}\right) - 2\,\frac{\sqrt{m_1 m_1' T_1 T_1'}}{m_2'}\cos\theta \qquad \text{--- (35)}$$

(35) is referred to as the Q-equation, and has many applications in nuclear physics.

REMARK

(1) When $Q = 0$ and all the particles are identical ($m_1 = m_1' = m_2 = m_2'$) the conservation of energy gives

$$p_1'^2 + p_2'^2 = p_1^2$$

while from the conservation of momentum

$$\vec{p}_1 = \vec{p}_1' + \vec{p}_2'$$

we have

$$p_1^2 = p_1'^2 + p_2'^2 + 2\vec{p}_1' \cdot \vec{p}_2'$$

Combining these results, we find that

$$\vec{p}_1' \cdot \vec{p}_2' = 0$$

or $\vec{p}_1'$ is $\perp$ar to p_2'

Thus in the lab-frame, the two particles move at right angles after the collisions

(2) In a CAPTURE PROCESS in which a particle of mass m_1 and momentum $\vec{p}_1$ collides with a particle of mass m_2 at rest, and a single particle of mass M results

$$\vec{p}_1 = \vec{p}_M$$

and

$$Q = T_M - T_1$$

However

$$T_M = \frac{p_M^2}{2M} = \frac{p_1^2}{2M} = T_1\left(\frac{m_1}{M}\right)$$

$$T_1 = \frac{p_1^2}{2m_1}$$

$$\frac{p_1^2}{2m_1}\left(\frac{m_1}{M}\right)$$

$$\therefore \quad Q = -T_1\left\{1 - \frac{m_1}{M}\right\}$$

$-Q$:= Excitation energy of the resulting particle i.e excited energy of a compound nucleus in a nuclear reaction.

In most cases

$$M \approx m_1 + m_2$$

so that

$$Q = -T_1 \left\{ \frac{m_2}{m_1 + m_2} \right\} \quad (37)$$

(3) The opposite of a capture process, which is called an explosion occurs when a particle explodes or decays into two or more fragments. This takes place for example when a grenade explodes, when a particle decays into several other particles or when a nucleus undergoes fission.