BOUND COURSES NOTE BOOK

Course Title: ~~PROBABILITY II~~ ORDINARY DIFFERENTIAL EQUATION.

Course Code: ~~STA 211~~ MAT 241

Course Outline:

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apters: COURSE OUTLINE.

1. Introduction
2. Derivation of O.D.E
3. Techniques for Solving first Order Linear & Non-linear differential equations.
4. Techniques for solving Order Two Homogeneous and non-homogeneous differential equations.
5. Technique for Solving nth order differential equations.
6. Difference and finite difference differential equation.
7. Error and Interpolations.
8. Numerical Integration and Numerical Differentiation.

COURSE MATERIAL: Advance Engineering Mathematics K.A stroud & Kracys.

Ordinary Differential Equation and Application.
by E.O Ayoola.

Recommended text: ??

10-05-18

INTRODUCTION.

What is O-DE?

① Ordinary Differential Equation is an equation that involve the unknown dependent functions and their derivatives. It also involves the coefficient that are functions of the independent variable.

For a D.E

i. It must possess dependent variable which will be a function

ii — — independent variable which will also be a function.

iii — possess the derivatives of the dependent variable.

⇒ How to Identify the order of a given equation.

② Definition: **The order of a DE** is the highest derivative that appears in the differential Equation.

③ Degree of a differential Equation: The degree of a DE is the highest power that appear in the DE.

④ Linear D.E: A differential equation is said to be linear if there is no product of the dependent variable or function and its derivatives, and neither the function or its derivatives appear to any power other than first power.

5. Non-linear D.E: If a DE is not linear, then it is called non-linear.

Examples:

① $\frac{dy}{dx} = \cos x$ is an e.g of D.E

Variable 'y' is the dependent variable, 'x' is independent. The variable that appear at the numerator is dependent while the denominator is always independent.

This example given is of order 1 and also Linear, also of degree 1.

② $\frac{d^2y}{dx^2} + ky^2 = 0$.

Y → is dependent.

X → independent.

It is linear

It is of order 2.

It is of degree 1.

③ $\left(\frac{d^2y}{dx^2}\right)^2 + Ky^2 = \sin x$.

* Y – dependent

X – Independent.

Order 2

Degree 2.

Non-Linear.

④. $\frac{d^2w}{dx^2} - xy\frac{dw}{dx} + w = 0$.

w – dependent

x – Independent.

Linear

Order 2

Degree 1.

⑤ $\left(\frac{d^2w}{dx^2}\right)^3 - xy\left(\frac{dw}{dx}\right)^2 + w = 0$.

* Non-linear

* Order 2

* Degree 3.

All the examples above are ODE, but the equation

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0}$$

is called a partial D·E (PDE) because it has one *DV but more than 1 IV*.

Any Diff equation with 1 D·V and more than 1 I·V are called **P·D·E.**

Generally, the equation of the form $F(x, y, y', y'', \dots y^{(n)}) = 0$ is called an nth order O·D·E.

11-05-18.

DERIVATION OF D·E

Just as a function can be obtained as a solution on any given D·E, the reverse process can also be carried out by obtaining a D·E from a given function through the process of eliminating the constant that appear in ~~Soft~~ such function.

Example: Form a D·E given the following functions.

i. $y = Ax$

$\frac{dy}{dx} = y' = A$.

$y = y'x$

$\therefore y' = \frac{y}{x}$

$\frac{dy}{dx} = \frac{y}{x}$.

② $y = Ax + B$.

$y' = A$. $\quad y'' = 0$.

$y = y'x + B$

$y = y'x + B$.

$\cancel{y-y}$

$y' = y''x + y'$ $\quad \cancel{x' = y''x}$

$y''x = 0$. $\quad y - y'x = B$

$$\frac{d^2y}{dx^2}x = 0. \qquad y - \cancel{y'x + y - y'x}$$

$y' =$

$y'' = 0$.

③. $y = A\cos ax + B\sin ax$.

$y' = -Aa\sin ax + Ba\cos ax$.

$y'' = -Aa^2\cos ax - Ba^2\sin ax$.

$y'' = -a^2(A\cos ax + B\sin ax)$.

$y'' = -a^2y$.

$y'' + a^2y = 0$.

$$\frac{d^2y}{dx^2} + a^2y = 0.$$

Assignment:

1. $Y' = X + \frac{A}{X}$
2. $Y = AX^2 + BX$.

3 Chapter.

⇒ Technique For Solving Order 1 Linear & Non-Linear D.E.

* Given a first order differential equation of the form $F(X, Y, Y') = 0$, which can also be written as $a_0\frac{dy}{dx} + a_1 y + a_2 = 0$ —*

Where a_0, a_1 & a_2 are either constant or a function. If a_0 ~~and~~ or a_1 are functions of the dependent variable then the equation * is a non-linear differential equation. but if a_0 or a_1 or both are functions of independent variable then the star * equation is a linear equation.

⇒ The techniques are classified into 5 namely:

1. Seperation of variable. (Variable seperable).
2. The homogeneous ~~style~~ type
3. The exact type.
4. The non-exact type.

These first-four takes care of linear differential equations.

5. Bernoulli equation. takes care of non-linear D.E.

1. SEPERATION OF VARIABLE:

⇒ If a D.E is of the form $\frac{dy}{dx} = f(x)g(y)$

then we can write the equation as

$$\frac{dy}{g(y)} = \cancel{\frac{f(x,y)}{dx}}\; f(x)dx. \qquad f(x,y)$$

Example:

* Solve the D.E $\frac{dy}{dx} = 2xy^2$.

Soln.

$$\frac{dy}{y^2} = 2x\,dx.$$

③ $\int \frac{dy}{y^2} = \int 2x\,dx$. $\frac{n+1}{n+1}$

$\int \frac{1}{y^2} dy = 2\int x\,dx$.

$\int y^{-2} dy = 2\int x\,dx$.

$\frac{-y^{-1}}{1} = \frac{2x^2}{2} + c$.

$\frac{-1}{y} = x^2 + c$.

$-1 = y(x^2+c)$.

$y = \frac{-1}{x^2+c}$.

② Solve the DE $(x+1)\frac{dy}{dx} = 2y$.

$(x+1)dy = 2y\,dx$.

$\int \frac{dy}{2y} = \int \frac{dx}{x+1}$

$\frac{1}{2}\int \frac{dy}{y} = \int \frac{1}{x+1} dx$.

$\frac{1}{2}\ln y = \ln(x+1) + \ln c$

$\ln y = 2\ln(x+1) + \ln c$

$\ln y = 2\ln c(x+1)$.

$\ln y = \ln[c(x+1)]^2$.

$y = [c(x+1)]^2$. Take the exponent of both sides.

$y = c^2(x+1)^2$.

④ $\frac{dy}{dx} = \frac{1+y}{2+x}$. (Try as an exercise)

2- HOMOGENEOUS TYPE:

A D.E of the form $\frac{dy}{dx} = F(x,y)$ is

said to be homogeneous of degree 'n' where $n \in \mathbb{Z}_+$ (n is a member of the $\mathbb{Z}$ of the function $F(tx, ty) = t^n f(x,y)$ where t is real number ($\mathbb{R}$) $t \in \mathbb{R}$.

Example.

1. $\frac{dy}{dx} = 2y^3 e^{(y/x)} - \frac{x^4}{x+3y}$, determine if the DE is homogeneous or not.

$F(x,y) = 2y^3 e^{(y/x)} - \frac{x^4}{x+3y}$.

$F(tx,ty) = 2t^3y^3 e^{(ty/tx)} - \frac{t^4x^4}{tx+3ty}$.

$= 2t^3y^3 e^{(y/x)} - \frac{t^3x^4}{x+3y}$.

$= t^3\left[2y^3 e^{(y/x)} - \frac{x^4}{x+3y}\right]$.

$= t^3 f(x,y)$.

∴ Therefore the DE given is an homogeneous type of degree 3 in x and y.

② $\frac{dy}{dx} = x^3y^2 - 3x^5$, $F(x,y) = x^3y^2 - 3x^5$.

$F(tx,ty) = t^3x^2t^2y^2 - 3t^5x^5$.

$= t^5x^2y^2 - 3t^5x^5$

$=$ ~~$t(t^5x^2y^2 - 3x)$~~.

$= t^5(x^2y^2 - 3x^5)$.

$= t^5 f(x,y)$.

→ Therefore, the DE given is an homogeneous type of degree 5 in x and y.

⇒ The solution of homogeneous type of D.E required that introduction of a new variable with the purpose of reducing the homogeneous type into the variable seperable type.

e.g Solve the D.E $y' = \frac{x^2+y^2}{xy}$.

$y' = \frac{t^2x^2+t^2y^2}{tx\,ty} = \frac{t^2(x^2+y^2)}{t^2xy}$

$y = t^0 f(x,y)$

So let $\boxed{v = y/x}$

$\boxed{y = vx}$

$y' = \frac{dy}{dx} = v'x + vx'$

But $y' = \frac{x^2+y^2}{xy} = v'x + vx'$

$\frac{x^2+y^2}{xy} = v'x + vx'$

$\frac{x^2+v^2x^2}{xvx} = v'x + vx'$

$\frac{x^2(1+v^2)}{x^2v} = v'x + vx'$

$\frac{1+v^2}{v} = v'x + vx'$.

$\frac{1+v^2}{v} = v'x + v$.

$v'x = \frac{1+v^2}{v} - \frac{v}{1} = \frac{1+v^2-v^2}{v}$

$v'x = \frac{1}{v}$.

$\frac{1}{v} = x\frac{dv}{dx}$.

$dx = vx\,dv$

$\int \frac{dx}{x} = \int v\,dv$

~~$\ln x = \frac{v^2}{1} + c$~~.

$\int v\,dv = \int \frac{dx}{x}$.

$\frac{v^2}{2} = \ln x + c$.

$v^2 = 2(\ln x + c)$.

$v = \sqrt{2(\ln x + c)}$

$v = \frac{y}{x}$ or $y = vx$.

$\frac{y}{x} = \sqrt{2(\ln x + c)}$

$y = x\sqrt{2(\ln x + c)}$.

③ Solve the D·E

$$\frac{dy}{dx} = \frac{x+3y}{2x}.$$

Let $v = y/x$.

$y = vx$.

$y' = v'x + vx'$

$y' = v'x + v$.

$\frac{dy}{dx} = v'x + v = \frac{x+3y}{2x}$.

$v'x + v = \frac{x+3y}{2x}$.

$\frac{x+3vx}{2x} = v'x + v$.

$\frac{1+3v}{2} = v'x + v$.

$\frac{1+3v}{2} - \frac{v}{1} = v'x$.

$\frac{1+3v-2v}{2} = v'x$.

$\frac{1+v}{2} = v'x$.

$\frac{1+v}{2} = \frac{dvx}{dx}$

$(1+v)dx = 2dvx$.

$\frac{dx}{x} = \frac{2dv}{1+v}$.

$\frac{dx}{2x} = \frac{dv}{1+v}$.

$\frac{dv}{1+v} = \frac{dx}{2x}$.

$\int \frac{dv}{1+v} = \int \frac{dx}{2x}$.

$\ln(1+v) = \frac{1}{2}\ln x + \ln c$.

$\ln(1+v) = \ln x^{1/2} + \ln c$.

$\ln(1+v) = \ln cx^{1/2}$.

$1+v = cx^{1/2}$.

$v = cx^{1/2} - 1$.

$\frac{y}{x} = cx^{1/2} - 1$

$y = x(cx^{1/2} - 1)$.

$y = cx^{3/2} - x$.

Exercise:

① $(x^2+xy)\frac{dy}{dx} = xy - y^2$

② $(x-y)\frac{dy}{dx} = x+y$.

SOLUTION.

3. $\frac{dy}{dx} = \frac{1+y}{2+x}$

$(2+x)dy = dx(1+y)$.

$\int \frac{dy}{1+y} = \int \frac{dx}{2+x}$.

$\ln(1+y) = \ln(2+x) + \ln c$.

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$\ln(1+y) = \ln c(2+x)$.

Taking exponents of both sides.

$e^{\ln(1+y)} = e^{\ln c(2+x)}$.

$(1+y) = c(2+x)$.

$y = c(2+x) - 1$

$y = 2c + cx - 1$.

② $(x^2+xy)\frac{dy}{dx} = xy - y^2$.

$$\frac{dy}{dx} = \frac{xy-y^2}{x^2+xy}$$

It is homogeneous of zero (0) degree. ~~order~~

$F(tx,ty) = t^n f(x,y)$.

$$= \frac{txty - t^2y^2}{t^2x^2 + txty} = \frac{t^2xy - t^2y^2}{t^2x^2+t^2xy}.$$

$$= \frac{\cancel{t^2}(xy-y^2)}{\cancel{t^2}(x^2+xy)}.$$

$= t^0 f(x,y)$.

So, Let $v = \frac{y}{x}$:

$y = vx$.

$$\frac{dy}{dx} = v'x + vx'$$

$= v'x + v$.

But $v'x + v = \frac{xy-y^2}{x^2+xy}$.

Also, $y = vx$.

$$v'x + v = \frac{x(vx) - (vx)^2}{x^2 + x(vx)} = \frac{vx^2 - v^2x^2}{x^2+vx^2}.$$

$$v'x + v = \frac{v\cancel{x}(x - vx)}{x^{\cancel{2}}(1+v)} = \frac{v\cancel{x^2}(1-v)}{\cancel{x^2}(1+v)}$$

$v'x = ~~\frac{v(x-vx)}{x(1+v)}~~ - \frac{v}{1}$, $\frac{v-v^2}{1+v} - \frac{v}{1}$

$$v'x = \frac{vx - v^2x - v[x+vx]}{x(1+v)} = \frac{vx - v^2x - \cancel{vx} - v^2x}{x+vx}.$$

$$v'x = \frac{-2v^2\cancel{x}}{\cancel{x}(1+v)}$$

$$x\frac{dv}{dx} = \frac{-2v^2}{1+v}.$$

$x(1+v)dv = -2v^2dx$

$$\int\left(\frac{1+v}{-2v^2}\right)dv = \int\frac{dx}{x}.$$

$$v'x = \frac{v - v^2 - v(1+v)}{1+v}.$$

$$v'x = \frac{\cancel{v} - v^2 - \cancel{v} - v^2}{1+v} = \frac{-2v^2}{1+v}.$$

$$\left(\frac{dv}{dx}\right)x = \frac{-2v^2}{1+v}.$$

$$\frac{dv(1+v)x}{-2v^2} = \frac{-2v^2dx}{-2v^2}.$$

$$\int dv\left(\frac{1+v}{-2v^2}\right) = \int\frac{dx}{x}$$

$$\int\left[\frac{-1}{2v^2} - \frac{v}{2v^2}\right]dv = \int\frac{dx}{x}.$$

$$-\frac{1}{2}\int\left[\frac{1}{v^2} + \frac{1}{v}\right]dv = \int\frac{dx}{x}$$

$$-\frac{1}{2}\left[-v^{-1} + \ln v\right] = \ln x + c.$$

$$-\frac{1}{2}\left[\frac{-1}{v}+\ln v\right]=\ln x+c \qquad \frac{1}{2v}$$

$$\frac{1}{2v}-\frac{1}{2}\ln v=\ln x+\ln A$$

$$\frac{1}{2v}-\frac{1}{2}\ln v=\cancel{2(\ln x+c)}\ \ln x+c$$

$$\frac{1}{2v}-c=\ln x+\ln v^{1/2}$$

$$2v^{-1}-c=\ln xv^{1/2}$$

$$2\left(\frac{y}{x}\right)^{-1}-c=\ln x\left(\frac{y}{x}\right)^{1/2}$$

$$\frac{2x}{y}-c=\ln$$

$$2v^{-1}-\ln v^{1/2}=\ln x+\ln A$$

$$2v^{-1}=\ln x+\ln A+\ln v^{1/2}$$

$$2v^{-1}=\ln xAv^{1/2}$$

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① $y=x+\frac{A}{x}$ $\quad Ax^{-1}$

$$y'=1-Ax^{-2}$$

$$y'=1-Ax^{-2}$$

$$y'-1=-Ax^{-2}$$

$$1-y'=Ax^{-2}$$

$$A=\frac{1-y'}{x^{-2}}=x^2(1-y')$$

$$y=x+\frac{x^{2}(1-y')}{x}$$

$$y=x+x(1-y')$$

$$y=x\left(1+1-\frac{dy}{dx}\right)$$

$$y=2x-x\frac{dy}{dx}$$

$$y+x\frac{dy}{dx}=2x$$

$$y+x\frac{dy}{dx}-2x=0$$

② $y=Ax^2+Bx$

$$y'=2Ax+B$$

$$y''=2A$$

$$\frac{y''}{2}=A$$

$$y'=y''x+B$$

$$y'-y''x=B$$

$$y=\frac{y''x^2}{2}+(y'-y''x)x$$

$$y=\frac{y''x^2}{2}+y'x-y''x^2$$

$$y=y'x-\frac{y''x^2}{2}$$

$$y=\frac{dy}{dx}x-\frac{d^2y}{dx^2}(x^2/2)$$

$$y=\frac{dy}{dx}(x)-\frac{d^2y}{2dx^2}x^2$$

Solution to Exercise 4 :

① $(x^2+xy)\frac{dy}{dx} = xy - y^2.$

$$\frac{dy}{dx} = \frac{xy-y^2}{x^2+xy}.$$

Let $v = y/x$.

$y = vx$.

$y' = v'x + v$.

$$v'x + v = \frac{x(vx) - v^2x^2}{x^2 + x(vx)}.$$

$$v'x+v = \frac{vx^2 - v^2x^2}{x^2+vx^2} = \frac{x^2(v-v^2)}{x^2(1+v)}.$$

$$v'x+v = \frac{v-v^2}{1+v}$$

~~$v'x+v$~~ $v'x = \frac{v-v^2}{1+v} - \frac{v}{1}.$

$$v'x = \frac{v-v^2-v(1+v)}{1+v} = \frac{\cancel{v}-v^2-\cancel{v}-v^2}{1+v}.$$

$$v'x = \frac{-2v^2}{1+v}.$$

$$x\frac{dv}{dx} = \frac{-2v^2}{1+v}.$$

$$\int \frac{1+v}{-2v^2}\,dv = \int\frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2}\left[\int \frac{1}{v^2}dv + \int\frac{v}{v^2}dv\right] = \int\frac{dx}{x}.$$

$$-\frac{1}{2}\left[\int v^{-2}dv + \int\frac{1}{v}dv\right] = \int\frac{dx}{x}.$$

$$-\frac{1}{2}\left[\frac{-1}{v}+\ln v\right] = \ln x + c.$$

$$\frac{1}{2v} - \frac{\ln v}{2} = \ln x + c.$$

$$\frac{1}{2}\left(\frac{1}{v} - \ln v\right) = \ln x + c$$

$$\frac{1}{v} - \ln v = 2\ln x + c.$$

$$\frac{1}{v} - \ln v = \ln x^2 + c.$$

$$\ln x^2 + \ln v = \frac{1}{v} - c.$$

$$-\ln x^2 - \ln v = -\frac{1}{v} + c.$$

$$\frac{1}{v} - c = \ln v + \ln x^2$$

$$\frac{1}{v} - c = \ln vx^2$$

But $v = y/x$.

$$\ln(y/x)x^2 = \frac{x}{y} - \ln A.$$

$$\ln xy = \frac{x}{y} - \ln A.$$

$$\ln xy + \ln A = \frac{x}{y}.$$

$$\ln Axy = \frac{x}{y}.$$

~~$Axy = e$~~

$$Axy = e^{x/y}.$$

$$xy = Ae^{x/y}.$$

⑤ $(x-y)\frac{dy}{dx} = x+y.$

$$\frac{dy}{dx} = \frac{x+y}{x-y}.$$

Let $v = y/x$.

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right).$$

$y = vx$.

$y' = v'x + v$.

$$v'x + v = \frac{x+y}{x-y}.$$

$$v'x + v = \frac{x+vx}{x-vx} = \frac{x(1+v)}{x(1-v)}$$

$$v'x + v = \frac{1+v}{1-v}$$

$$v'x = \frac{1+v}{1-v} - \frac{v}{1}.$$

$$v'x = \frac{1+v-v(1-v)}{1-v}$$

$$v'x = \frac{1+\cancel{v}-\cancel{v}+v^2}{1-v} = \frac{1+v^2}{1-v}$$

$$v'x = \frac{1+v^2}{1-v}.$$

$$x\left(\frac{dv}{dx}\right) = \frac{1+v^2}{1-v}$$

$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{dx}{x}.$$

$$\int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv = \int \frac{dx}{x}.$$

$$\tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln x + \ln A.$$

But $v = y/x$.

$y = vx$.

$$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln x + \ln A.$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln x + \ln A + \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right).$$

$$= \ln Ax\left(1+\frac{y^2}{x^2}\right)^{1/2}.$$

17-05-18.

EXACT TYPE.

The equation of the form $[M(x,y)dx + N(x,y)dy = 0]$ is called exact iff.

or $$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}.$$

The given equation will be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

① Example: Determine the exactness or otherwise of the D.E $y^2dx + 2xy\,dy = 0$.

$M(x,y) = y^2$

$N(x,y) = 2xy$

$$\frac{\partial M}{\partial y} = 2y. \qquad \frac{\partial N}{\partial x} = 2y.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2y$, therefore the equation $y^2dx + 2xy\,dy = 0$ is an exact type.

② Determine the exactness or otherwise of the D.E $(3x^2+4xy)dx + (2x^2+2y)dy = 0$.

$M(x,y) = 3x^2+4xy$

$N(x,y) = 2x^2+2y$.

So, $$\frac{\partial M}{\partial y} = 4x.$$

$$\frac{\partial N}{\partial x} = 4x.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x$;

∴ The equation $(3x^2+4xy)dx + (2x^2+2y)dy=0$ is an exact type.

Note: M is always the coefficient of dx while that of N is for dy.

How to Solve Equation of Exact Type.

Solve the D.E $(3x^2+4xy)dx + (2x^2+2y)dy=0$.

Solution.

$$(3x^2+4xy)dx + (2x^2+2y)dy = 0.$$

$$M(x,y) = \frac{\partial F}{\partial x} = 3x^2+4xy.$$

$F = \int (3x^2+4xy)\,dx \longrightarrow$ Phi

$$F = x^3 + 2x^2y + \Phi(y).$$

But $N(x,y) = \frac{\partial F}{\partial y} = 2x^2+2y$

~~But~~ Also $\frac{\partial F}{\partial y} = 2x^2 + \Phi'(y)$.

$$\therefore\ 2x^2+2y = 2x^2+\Phi'(y).$$

$$2y = \Phi'(y).$$

$$\int 2y\,dy = \int \Phi'(y)\,dy.$$

$$\Phi(y) = y^2.$$

So ∴ $F = x^3+2x^2y+y^2$.

I need to solve more questions on exact type.

② Solve the D.E $(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y)dy = 0$.

~~Test the exactness:~~

Solution.

$$M(x,y) = 2x\cos y + 3x^2y.$$

$$N(x,y) = x^3 - x^2\sin y.$$

$$M(x,y) = \frac{\partial F}{\partial x} = 2x\cos y + 3x^2y.$$

$$F = \int (2x\cos y + 3x^2y)\,dx.$$

~~$F = x + x^3$~~

$$F = x^2\cos y + x^3y + \Phi(y).$$

Recall $N(x,y) = \frac{\partial F}{\partial y} = x^3 - x^2\sin y$.

But $\frac{\partial F}{\partial y} = -x^2\sin y + x^3 + \Phi'(y)$.

So $x^3 - x^2\sin y = -x^2\sin y + x^3 + \Phi'(y)$

$$\Phi'(y) = 0.$$

Taking integral of both sides.

$$\Phi(y) = \int 0\,dy.$$

$$\Phi(y) = 0.$$

∴ $\boxed{F = x^2\cos y + x^3y.}$ or

$$F = x^2\cos y + x^3y + C.$$

③ $(3x^2y - 6x)dx + (x^3+2y)dy = 0$.

Soln.

$$M(x,y) = 3x^2y - 6x$$

$$N(x,y) = x^3+2y$$

How to find I·F for non-exact equation?? .

$N(x,y) = \frac{\partial F}{\partial y} = x^3 + 2y$

$M(x,y) = \frac{\partial F}{\partial x} = 3x^2y - 6x$

$F = \int (x^3 + 2y)\, dy$

$F = x^3y + y^2 + \Phi(x)$.

$M(x,y) = \frac{\partial F}{\partial x} = 3x^2y - 6x$.

Differentiating F with respect to x.

$\frac{\partial F}{\partial x} = 3x^2y + \Phi'(x)$.

So, $3x^2y + \Phi'(x) = 3x^2y - 6x$.

$\Phi'(x) = -6x$.

Integrating both sides to get $\Phi(x)$.

$\therefore \Phi(x) = \int -6x\, dx$

$= \frac{-6x^2}{2} = -3x^2$.

$\therefore$ ~~F = x^3y~~

$F = x^3y + y^2 - 3x^2$.

18-05-18. NON-EXACT TYPE.

Consider $y\,dx + (x^2y - x)dy = 0$

$M = y \qquad \frac{\partial M}{\partial y} = 1$.

$N = x^2y - x, \quad \frac{\partial N}{\partial x} = 2xy - 1$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, Hence the given equation is a non-exact type.

If we multiply the given equation by $\frac{1}{x^2}$, we will get

$\frac{y}{x^2} dx + (y - \frac{1}{x}) dy = 0$.

$M = \frac{y}{x^2} \qquad \frac{\partial M}{\partial y} = \frac{1}{x^2}$.

$N = y - \frac{1}{x} \qquad \frac{\partial N}{\partial x} = +\frac{1}{x^2}$.

The function introduced to the non-exact equation to make it an exact equation is an integrating factor.

In this example, I·F is $\frac{1}{x^2}$.

Consider the D E of the form

$\boxed{\frac{dy}{dx} + Py = Q}$ ..... (2.6).

Where P & Q are either functions of independent variables x or constants.

Eqn (2.6) is not exact but we can make it exact if we introduce integrating factor I·F which is obtained as:

$I \cdot F = e^{\int P dx}$.

Example: Solve the D.E

$\frac{dy}{dx} + 3y = e^{2x}$.

$P = 3, \; Q = e^{2x}$.

$I \cdot F = e^{\int 3dx} = e^{3x}$

Multiply the given D·E by I·F.

$e^{3x}\left(\frac{dy}{dx}\right) + 3e^{3x}y = e^{5x} \; \ldots *$

Observe that the L·H·S of ~~this~~ equ * is ~~the~~ differential of ye^{3x}.

i·e $\frac{d}{dx}(ye^{3x}) = $ L·H·S of equ *.

$\frac{d}{dx}(ye^{3x}) = e^{3x}\frac{dy}{dx} + 3e^{3x}y$.

$\frac{d}{dx}(ye^{3x}) = e^{5x}$

$d(ye^{3x}) = e^{5x}dx$.

$ye^{3x} = \int e^{5x}dx$.

$ye^{3x} = \frac{e^{5x}}{5} + C$.

$y = \frac{e^{2x}}{5} + Ce^{-3x}$.

② Solve the D·E $y' - y = x$.

Soln.

$P = -1, \; Q = x$.

$I \cdot F = e^{\int -1dx} = e^{-x}$

$e^{-x}\frac{dy}{dx} - ye^{-x} = xe^{-x} \; \cdot\cdot *$

Observe that L·H·S of * is the differential of

$\frac{d}{dx}(ye^{-x}) = $ L·H·S of this eqn.

LOPET.

$\frac{d}{dx}(ye^{-x}) = xe^{-x}$

$d(ye^{-x}) = xe^{-x}dx$.

$ye^{-x} = \int xe^{-x}dx$.

Integrate the R·H·S by parts.

$\int xe^{-x}dx = \int u\,dv = uv - \int v\,du$.

$u = x, \quad dv = e^{-x}dx$.

$du = dx. \quad v = \int e^{-x}dx$

$v = -e^{-x}$

$\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx$.

$= -xe^{-x} + \int e^{-x}dx$.

$= -xe^{-x} - e^{-x} + c$.

∴ ~~$ye^{-x} = e^{-x}(-x$~~

$ye^{-x} = -e^{-x}(x+1) + c$.

$y = -(x+1) + ce^{x}$.

③ Solve the D·E $\frac{dy}{dx} + y\cot x = \cos x$.

$P = \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$.

$I \cdot F = e^{\int \cot x dx} = e^{\int \frac{\cos x}{\sin x}dx}$

$I \cdot F = e^{\ln \sin x} = \sin x$.

$I \cdot F = \sin x$.

Multiply through by $\sin x$.

$\sin x\left(\frac{dy}{dx}\right) + y\cos x = \sin x\cos x$

$\frac{d}{dx}(y\sin x) = \sin x\cos x$.

for Bernoulli equation, Sub $Z = y^{1-n}$

$y \sin x = \int \cos x \sin x \, dx.$

$y \sin x = \frac{\sin^2 x}{2} + c$

$y = \frac{\sin x}{2} + c \operatorname{cosec} x.$

BERNOULLI EQUATION.

Non-Linear I order. D.E

The equation of the form $\frac{dy}{dx} + P(x)y = q(x)y^n$. *

We introduce $z = y^{1-n}$ ---- (2.9).

$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ ---- (2.10).

Eg 1. Solve the D.E $\frac{dy}{dx} + \frac{y}{x} = xy^2.$

$n = 2.$

Divide through by y^n which in this case y^2 since $n = 2$.

$y^{-2}\frac{dy}{dx} + \frac{1}{x}.y^{-1} = x.$

But $z = y^{1-n} = y^{1-2} = y^{-1}.$

$Z = y^{-1}.$ $\frac{1}{y} = z.$ $zy = 1$ $z = \frac{1}{y}.$ $y = \frac{1}{z}$

$\frac{dz}{dx} = -1y^{-2}\frac{dy}{dx} = -y^{-2}\frac{dy}{dx}$ ...2.11

$-\frac{dz}{dx} + \frac{Z}{x} = x.$

$-\frac{dz}{dx} + Zx^{-1} = x.$

Multiply through by -1

$\frac{dz}{dx} - zx^{-1} = -x.$ ... 2.12

Equ (2.12) is now linear.

So $P = -\frac{1}{x}.$

$I.F = e^{\int P dx} = e^{\int -\frac{1}{x}dx}$

$= e^{-\ln x} = e^{\ln x^{-1}}$

$I.F = x^{-1}$

$(x^{-1})\frac{dz}{dx} - Zx^{-2} = -x \times \frac{1}{x} = -1$

$(x^{-1})\frac{dz}{dx} - \frac{z}{x^2} = -1.$

$\frac{d}{dx}(zx^{-1}) = -1.$

$zx^{-1} = \int -1 \, dx.$

$zx^{-1} = -x + c.$

$\frac{z}{x} = -x + c$

$z = -x^2 + cx.$

But $z = y^{-1}$

$\frac{1}{y} = -x^2 + Cx.$

$y(-x^2 + Cx) = 1.$

$y = \frac{1}{-x^2 + Cx} = \frac{1}{-x(x-c)}.$

② Solve the D.E $x^2y - x^3\frac{dy}{dx} = y^4 \cos x.$

Soln-

Rearrange:

$$-x^3\frac{dy}{dx} + x^2y = y^4\cos x.$$

$$\frac{dy}{dx} - \frac{y}{x} = -\frac{y^4}{x^3}\cos x.$$

Where n=4.

$$y^{-4}\frac{dy}{dx} - \frac{\cancel{x}y^{-3}}{x} = -\frac{\cos x}{x^3} \quad \ldots ③$$

Let $z = y^{-3}$.

$$\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx} \quad \ldots ④$$

$$-\frac{1}{3}\frac{dz}{dx} - \frac{z}{x} = -\frac{\cos x}{x^3}.$$

$$-\frac{1}{3}\frac{dz}{dx} - zx^{-1} = -\frac{\cos x}{x^3}.$$

~~$P = -x^{-1}$~~, $I{\cdot}F = e^{\int Pdx} = e^{\int -\frac{1}{x}dx}$.

I·F =

Multiply through by -3.

$$\frac{dz}{dx} + 3zx^{-1} = \frac{3\cos x}{x^3}.$$

$P = 3x^{-1}$, $I{\cdot}F = e^{\int 3x^{-1}dx}$

$$I{\cdot}F = e^{3\int\frac{1}{x}dx} = e^{3\ln x} = e^{\ln x^3}.$$

$I{\cdot}F = x^3$.

Multiply through by I·F.

$$x^3\frac{dz}{dx} + 3zx^2 = 3\cos x.$$

$$\frac{d}{dx}(zx^3) = 3\cos x$$

or

$$zx^3 = \int 3\cos x\,dx.$$

$$zx^3 = 3\int\cos x\,dx = 3\sin x + c.$$

$$z = \frac{3\sin x + C}{x^3} = \frac{3\sin x}{x^3} + \frac{C}{x^3}.$$

$$\frac{1}{y^3} = \frac{3\sin x + c}{x^3}$$

$$y^3 = \frac{x^3}{3\sin x + C}.$$

$$y = \sqrt[3]{\frac{x^3}{3\sin x + c}} = \left(\frac{x^3}{3\sin x + C}\right)^{1/3}.$$

24-05-18. SECOND ORDER.

SECOND ORDER NON-LINEAR EQUATIONS.

$F(x,y,y',y'') = 0$.

Case 1: When x is missing

i.e $F(y,y',y'') = 0$

Case 2: When y is missing

i.e $F(x,y',y'') = 0$.

Case 2: If y is missing from

$F(x,y,y',y'') = 0$

$G(x,y',y'') = 0 \quad \ldots\ldots (*)$

Suppose y is the solution of equation (*),

We can write:

$$\frac{dy}{dx} = V = y'$$

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$$\frac{d^2y}{dx^2} = y'' = v'$$

Putting the two equations above in equ (*), we have $G(x, v, v'') = 0$.

Example: Solve the D.E $x\frac{d^2y}{dx^2} = 2\left[\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx}\right]$.

Soln.

Let $v = \frac{dy}{dx} = y'$, $y'' = v'$

$$x\frac{dv}{dx} = 2\left[v^2 - v\right]$$

$$x\frac{dv}{dx} = 2(v^2 - v).$$

$$x\,dv = 2(v^2 - v)\,dx.$$

$$\int \frac{dv}{v^2 - v} = \int \frac{2\,dx}{x}.$$

$$\int \frac{dv}{v(v-1)} = 2\int \frac{dx}{x}$$

$$\frac{1}{v(v-1)} = \frac{A}{v} + \frac{B}{v-1}$$

$$1 = A(v-1) + Bv.$$

When $v = 1$

$1 = B(1)$

$B = 1$

When $v = 0$

$1 = -A$

$A = -1$

$$\int\left(\frac{1}{v-1} - \frac{1}{v}\right)dv = 2\int \frac{dx}{x}.$$ Ans: $y =$.

$$\ln(v-1) - \ln v = 2\ln x + c.$$

$$\ln\left(\frac{v-1}{v}\right) = \ln x^2 + \ln c.$$

$$\ln\left(\frac{v-1}{v}\right) = \ln cx^2$$

$$\frac{v-1}{v} = cx^2.$$

$$1 - \frac{1}{v} = cx^2$$

$$v = \frac{1}{1 - cx^2}.$$

$-\frac{1}{v} = cx^2 - 1$

$\frac{1}{v} = 1 - cx^2.$

$v(1 - cx^2) = 1$

$v = \frac{1}{1 - cx^2}$

But $v = y' = \frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{1 - cx^2}$$

$$\int dy = \int\left(\frac{1}{1 - cx^2}\right)dx.$$

$$y = \int \frac{1}{1 - cx^2}\,dx.$$ complete. If $c = +$ and if $c = 0$

Ans: $y = \frac{1}{2a}\ln\left(\frac{1 + ax}{1 - ax}\right)$ if c is $+$.

If $c > 0$ i.e $c = a$.

Case 2: If x is missing in $F(x, y, y', y'') = 0$.

$G(y, y', y'') = 0.$

Let $v = \frac{dy}{dx} = y'$, $\frac{dv}{dx} = \frac{d^2y}{dx^2} = v'$

But $\frac{dv}{dy}\cdot\frac{dy}{dx} = \cancel{v\frac{dy}{dx}} = v\frac{dv}{dy} = \frac{d^2y}{dx^2}$

$$\frac{dv}{dy}\cdot\frac{dy}{dx} = v\frac{dv}{dy}.$$

So, $G\left(y, v, v\frac{dv}{dy}\right) = 0$.

Example:

$$y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx}.$$

Soln.

Let $v = \frac{dy}{dx} = y'$, $y'' = \frac{d^2y}{dx^2} = v\frac{dv}{dy}$.

$$yv\frac{dv}{dy} = v^2 + 2v$$

$$y\frac{dv}{dy} = \frac{v^2+2v}{v} = \frac{v(v+2)}{v}$$

$$y\frac{dv}{dy} = v+2.$$

$$y\,dv = (v+2)\,dy.$$

$$\int \frac{dv}{v+2} = \int \frac{dy}{y}.$$

$$\ln(v+2) = \ln y + \ln C.$$

$$\ln(v+2) = \ln Cy.$$

$$v+2 = Cy.$$

$$v = Cy-2.$$

$$\frac{dy}{dx} = Cy-2.$$

$$\int \frac{dy}{Cy-2} = \int dx$$

$$\frac{1}{c}\ln(cy-2) = x + k.$$

$$\ln(cy-2) = cx + A.$$

$$Cy - 2 = e^{cx+A}$$

$$Cy = e^{cx+A} + 2.$$

$$y = \frac{Be^{cx}+2}{c}.$$

N.B

$B = e^{A}$.

25-05-18. SECOND ORDER LINEAR HOMOGENEOUS &NON EQUATION

The equation $F(x, y, y', y'') = 0$.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = f(x) \quad \ldots (i)$$

a_0, a_1, a_2 are either functions of x or constant.

If $f(x) = 0$ in equ (i), then equ (i) is referred to as homogeneous linear second order differential equation. On the other hand if $f(x) \neq 0$ in equ (i), then equ (i) is referred to as non-homogeneous linear second order differential equation. If $f(x) \neq 0$, $f(x)$ can either be a function of x or a constant of non-homogeneous linear second order differential equation.

If $y_1(x)$ and $y_2(x)$ are solution of equ (i), So also is $\boxed{y(x) = y_1(x) + y_2(x)}$ a solution.

From equ (i), If $a_0 \equiv 0$, then equ (i) is order 1.

i-e $a_1(x)\frac{dy}{dx} + a_2(x)y = 0$.

$$\boxed{y = Ae^{m_1x} + Be^{m_2x}}$$

$$y = e^{mx}(A_1 + \underset{A_2}{B}x).$$

$$a_1(x)\frac{dy}{dx} = -a_2(x)y.$$

Let assume a_1 & a_2 are constant.

$$\therefore a_1\frac{dy}{dx} = -a_2y.$$

$$\frac{dy}{dx} = \frac{-a_2}{a_1}y.$$

Let $\frac{-a_2}{a_1} = k$. or m

So, $\frac{dy}{dx} = ky$.

$$\int\frac{dy}{y} = \int k\,dx$$

$$\ln y = kx + c$$

$$y = e^{kx+c} = e^{kx}.e^{c}$$

$$\boxed{y = Ae^{kx} = Ae^{mx}}*$$

$$y' = Ake^{kx} = Ame^{mx}.$$

$$y'' = Am^2e^{mx}$$

Subs. y, y' and y'' in equ (i).

$$\therefore a_0Am^2e^{mx} + a_1Ame^{mx} + a_2Ae^{mx} = 0. \quad ②$$

$$Ae^{mx}[a_0m^2 + a_1m + a_2] = 0.$$

$a_0m^2 + a_1m + a_2 = 0 \rightarrow$ is a particular ...... ③ quadratic equation in m. This equation is called auxilliary equation or characteristics equation.

The solution of equation ③ is often appear in 3 forms.

$\therefore y = \alpha \pm j\beta \qquad y = e^{\alpha x}(A_1\cos\beta x + A_2\sin\beta x)$

i-e

① Real and Distinct roots.

② Real and equal roots.

③ Complex root.

1) Real and Distinct roots.

Eg: Solve $\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = 0.$

Obtain the auxilliary equation.

$$m^2 - 3m + 2 = 0.$$

$$m^2 - m - 2m + 2 = 0.$$

$$m(m-1) + 2(m-1) = 0.$$

$$(m-2)(m-1) = 0.$$

$$m_1 = 2 \text{ or } m_2 = 1.$$

$$m_1 = 1 \text{ \& } m_2 = 2.$$

Since the roots are real and distinct, the the general solution of the equation is

$y_1(x) = A_1e^{x}$ From $\left(y = Ae^{mx}.\right)$

$$y_2(x) = A_2e^{2x}$$

$$y(x) = y_1(x) + y_2(x)$$

$$y(x) = A_1e^{x} + A_2e^{2x}.$$

② $\frac{d^2y}{dx^2} = -4\frac{dy}{dx} + 5.$

Soln. $\frac{d^2y}{dx^2} + \frac{4dy}{dy} - 5 = 0.$

$$m^2 + 4m - 5 = 0$$

$m = -5$ or 1.

$y_1(x) = A_1 e^{m_1x}$, $y_2(x) = A_2 e^{m_2x}$

$y_1(x) = Ae^{-5x}$, $y_2(x) = A_2 e^{x}$

$\therefore y(x) = A_1 e^{-5x} + A_2 e^{x}$.

2) Real and Equal Roots:

① Solve: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$.

$m^2 - 6m + 9 = 0$.

$m_1 = 3$, $m_2 = 3$.

$y_1(x) = A_1 e^{3x}$, $y_2(x) = A_2 e^{3x}$.

$\therefore y(x) = A_1 e^{3x} + A_2 e^{3x}$.

$y(x) = e^{3x}(A_1 + A_2)$.

$y(x) = Ae^{3x}$ where $A = A_1 + A_2$.

For a equation with real & repeated roots, the general solution will be of the form: $y(x) = (A + Bx)e^{mx}$.

$\therefore y(x) = (A + Bx)e^{3x}$ is the general solution of the given equation.

② $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$.

Soln.

$m^2 + 4m + 4 = 0$.

$m_1 = -2$, $m_2 = -2$.

$y_1(x) = A_1 e^{-2x}$, $y_2(x) = A_2 e^{-2x}$.

$\therefore y(x) = A_1 e^{-2x} + A_2 e^{-2x}$

$= (A_1 + A_2)e^{-2x} = Ae^{-2x}$

$\therefore$ Since the auxiliary roots are repeated, then the general solution is

$y(x) = (A + Bx)e^{-2x}$.

③ Complex Conjugate Roots:

If the roots of $a_0m^2 + a_1m + a_2 = 0$ are $m_1 = \alpha + i\beta$ and $\alpha - i\beta = m_2$ where α & β are real numbers, then $y_1(x) = A_1 e^{m_1x} = A_1 e^{(\alpha+i\beta)x}$

$y_2(x) = A_2 e^{m_2x} = A_2 e^{(\alpha - i\beta)x}$.

So, the general solution will be

$y(x) = y_1(x) + y_2(x) = A_1 e^{(\alpha+i\beta)x} + A_2 e^{(\alpha - i\beta)x}$.

Chang to polar form to have:

$y(x) = A_1 e^{\alpha x + i\beta x} + A_2 e^{\alpha x - i\beta x}$

$= A_1 e^{\alpha x} \cdot e^{i\beta x} + A_2 e^{\alpha x} \cdot e^{-i\beta x}$.

$= e^{\alpha x}\left[Ae^{i\beta x} + A_2 e^{-i\beta x}\right]$.

Recall that $e^{i\theta} = (\cos\theta + i\sin\theta)$.

$\therefore e^{-i\theta} = \cos\theta - i\sin\theta$.

So $y(x) = e^{\alpha x}\left[A_1(\cos\beta x + i\sin\beta x) + A_2(\cos\beta x - i\sin\beta x)\right]$

$y(x) = e^{\alpha x}(A_1\cos\beta x + iA_1\sin\beta x + A_2\cos\beta x - iA_2\sin\beta x)$

$y(x) = e^{\alpha x}\left[A_1\cos\beta x + A_2\cos\beta x + iA_1\sin\beta x - iA_2\sin\beta x\right]$.

$y(x) = e^{\alpha x}\left[(A_1 + A_2)\cos\beta x + i(A_1 - A_2)\sin\beta x\right]$.

$y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.

Where $A = A_1 + A_2$ and $B = i(A_1 - A_2)$

Summarily,

- For real and distinct root, the general solution is of the form $y(x) = A_1e^{m_1x} + A_2e^{m_2x}$
- For real and repeated root, the general solution is of the form $y(x) = (A_1 + B_2x)e^{mx}$.
- For complex conjugate roots, the general solution is of the form

$$y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

Example:

Solve $y'' - 2y' + 10y = 0$.

Soln. $m^2 - 2m + 10 = 0$.

$m_1 = 1 + 3i$, $m_2 = 1 - 3i$.

$\alpha = 1$, $\beta = 3$.

The general solution will be

$y(x) = e^{x}(A\cos 3x + B\sin 3x)$.

$y'' + y = 0$.

$m^2 + 1 = 0$.

$m^2 = -1$

$m = \pm i$.

$\therefore m_1 = i$, $m_2 = -i$

Where

$\alpha = 0$, $\beta = 1$:

$\therefore y(x) = A\cos x + B\sin x$.

③ Solve the Initial value problem (IVP)

$y'' - 6y' + 25y = 0$.

$y(0) = -3$, $y'(0) = -1$.

Where $y(0)$ and $y'(0)$ are initial values.

Soln.

$m^2 - 6m + 25 = 0$.

$m_1 = 3 + 4i$, $m_2 = 3 - 4i$

Where $\alpha = 3$ and $\beta = 4$.

$\Rightarrow y(x) = e^{3x}(A\cos 4x + B\sin 4x)$.

Since $y(0) = -3 \rightarrow$ given

$\therefore -3 = e^0(A\cos 0 + B\sin 0)$:

$-3 = A$

$A = -3$.

$y'(x) = \overset{4x}{-}3Ae^{3x}\sin 4x + 3\times 4e^{3x}B\cos 4x$.

$= -12Ae^{3x}\sin 4x + 12e^{3x}_{B}\cos 4x$.

$= e^{3x}[(3A_1 - 4A_2)\sin 4x + (4A_1 + 3A_2)\cos 4x]$.

But $y'(0) = -1$.

~~$-1 = e^0([3(-3) - 4($~~

$-1 = e^0[(3A_1 - 4A_2)\sin 0 + (4A_1 + 3A_2)\cos 0]$.

~~$-1 = 4A_1 + 3B$~~.

$-1 = 3A + 4B$

$4B = 8$

$B = 2.$

$y'(0) = -1$

$y'(x) = e^{3x}[-12A\sin 4x + 12B\cos 4x].$

$y'(x) = 12e^{3x}[-A\sin 4x + B\cos 4x].$

$-1 = 12e^0(B\cos 0).$

$-1 = 12B$

$B = -1/12.$

The particular solution is

$y(x) = e^{3x}[-3\cos 4x + 2\sin 4x].$

31-05-18

SECOND ORDER NON-HOMOGENEOUS D-E.

If $f(x) \neq 0$ in equ (i); then the equ (i) is non-homogenuous.

The solution has two parts i.e the particular integral and the complementary function

* We shall solve this type of D·E with the following techniques namely:

* Method of Undetermined coefficient.
* Method of Variation of Parameter or Constant
* Operator D method *.

⇒ METHOD OF UNDERTMINED COEFFICIENT:

We shall consider when the function $f(x)$ is of the following form:

i. $P_n(x) = a_n x^n + a_n x^{n-1} + \dots a_0$

ii. $P_n(x)e^{ax} = (a_n x^n + a_{n-1}x^{n-1} + \dots + a_0)e^{ax}$

iii. $P_n(x) \times \sin\beta x = (a_n x^n + a_{n-1}x^{n-1} + \dots + a_0)\sin\beta x$

iv. $P_n(x)\cos\beta x = (a_n x^n + a_{n-1}x^{n-1} + \dots a_0)\cos\beta$

v. $P_n(x)\sinh x$

vi. $P_n(x)\cosh x.$

⇒ ~~We~~ ~~Summa~~

Example: Solve the D·E

$y'' - 5y' + 6y = x^2.$

Soln.

The given equation is non-homogeneous equat because $f(x) \neq 0$.

⇒ ~~Divide~~ Let $f(x) = 0$, i.e $x^2 = 0$, to make the given equation homogeneous.

⇒ $y'' - 5y' + 6y = 0.$

This has auxilliary equation

$m^2 - 5m + 6 = 0.$

Solve this to get

$m = 2, m = 3.$

∴ The general solution of the homogenuous part or complementary function is

$y(x) = A_1e^{2x} + A_2e^{3x}$

$y_c(x) = Ae^{2x} + Be^{3x}.$

The Solution $y(x) = Ae^{2x} + Be^{3x}$ is the complementary function.

i.e $y_c(x) = Ae^{2x} + Be^{3x}$.

To find the particular integral, we assume the general form of the RHS of the given problem which is a polynomial of order 2. So that

$$y_p(x) = Cx^2 + Dx + E.$$

$$y_p' = 2Cx + D.$$

$$y_p'' = 2C.$$

Substituting this onto the given equ, we have

$$\cancel{2C - 10Cx + 6}$$

$$2C - 5(2Cx + D) + 6(Cx^2 + Dx + E) = x^2.$$

$$2C - 10Cx - 5D + 6Cx^2 + 6Dx + 6E = x^2.$$

$$6Cx^2 + (6D - 10C)x + 2C - 5D + 6E = x^2.$$

Equate corresponding coefficient.

$$6C = 1$$

$$\Rightarrow C = 1/6$$

$$6D - 10C = 0$$

$$6D = 10/6$$

$$D = \frac{10}{36} = \frac{5}{18}.$$

$$2C - 5D + 6E = 0.$$

$$\frac{1}{3} - \frac{25}{18} + 6E = 0$$

$$\cancel{6E = -\left(\frac{4 - 25}{12}\right)} \qquad 6E = \frac{-6 + 25}{18} = \frac{19}{18}$$

$$\cancel{6E = -\left(\frac{-21}{12}\right)}$$

$$\cancel{6E = \frac{21}{12}}$$

$$\cancel{E = \frac{21\,7}{12 \times 6\,2} = \frac{7}{24}}.$$

$$E = \frac{19}{108}.$$

$$y_p(x) = \frac{1}{6}x^2 + \frac{5}{18}x + \frac{19}{108}$$

$$y(x) = y_c(x) + y_p(x).$$

$$y(x) = Ae^{2x} + Be^{3x} + \frac{1}{6}x^2 + \frac{5}{18}x + \frac{19}{108}.$$

Where A and B are the arbitrary constant.

Q: How to find the exact value?

A: A & B can be determined if the initial conditions are given.

② Solve the D.E

$$y'' + y' = xe^{2x}.$$

Soln.

Let $f(x) = 0$.

$$\Rightarrow y'' + y' = 0.$$

$$m^2 + 1 = 0.$$

$$m = \pm\sqrt{-1} = \pm i$$

$$m_1 = i, \quad m_2 = -i$$

So the compliment ary function is

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$m = \alpha \pm i\beta$.

$y_c(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.

$y_c(x) = e^{0}(A\cos x$.

$y_c(x) = A\cos x + B\sin x$.

But $f(x)$ is of the form $P_n(x)e^{ax}$, where $P_n(x)$ is a polynomial of degree 1, so the P.I is

$y_p(x) = e^{2x}(a+bx)$.

Differentiate P.I twice & substitute into the given D.E, we have

$y'_p(x) = e^{2x}(2a+2bx+b)$.

$y''_p(x) = e^{2x}(4a+4bx+4b)$

Let a and b equals C & D.

$y'_p(x) = e^{2x}(2C+2Cx+D)$.

$y''_p(x) = e^{2x}(2C) + 2(C+2Cx+2D)e^{2x}$

$= e^{2x}(2C+2C+4Cx+4D)$.

$= e^{2x}(4Cx+4C+4D)$.

Substitute into the given equation,

$e^{2x}(4Cx+4C+4D) + e^{2x}(Cx+D) = xe^{2x}$.

$e^{2x}(5Cx+4C+5D) = xe^{2x}$.

$5Cx+4C+5D = x$.

$5C = 1$.

$C = 1/5$.

$4C+5D = 0$.

$5D = -4C$

$D = -\frac{4}{5}\cdot\frac{1}{5} = -\frac{4}{25}$.

P.I =

$\Rightarrow y_p(x) = e^{2x}\left[\frac{1}{5}x - \frac{4}{25}\right]$.

∴ The general solution of the D.E is given as:

$y(x) = y_c(x) + y_p(x)$.

$y(x) = A\cos x + B\sin x + e^{2x}\left[\frac{x}{5} - \frac{4}{25}\right]$

$y(x) = A\cos x + B\sin x + e^{2x}\left(\frac{5x-4}{25}\right)$.

Observation:

- If $f(x)$ is of the form e^{2x} or $2e^{2x}$, then we will have constant.
- If $f(x)$ is $x\sin x$, $x^2\cos x$.

$(Cx+D)\sin x$.

01-06-18.

Solve $y'' - y = 2e^{x}$.

Soln.

Let $2e^{x} = 0$.

$\Rightarrow y'' - y = 0$.

$m^2 - 1 = 0$.

$m = \pm 1$.

$m_1 = 1$, $m_2 = -1$.

$y_c(x) = y_1(x) + y_2(x)$.

$y_1(x) = Ae^{x}$, $y_2(x) = Be^{-x}$.

∴ $y_c(x) = Ae^{x} + Be^{-x}$.

P.I, knowing that $f(x) = 2e^x$.

P.I : $Ce^x = y_p(x)$.

$y'_p = Ce^x$

$y''_p = Ce^x$.

$Ce^x - Ce^x = 2e^x$.

$\boxed{2e^x = 0}$.

o solution, this means that the choice of r P.I is not appropriate.

nstead of $y_p(x) = Ce^x$, we will write

$y_p(x) = Cxe^x$.

$y'_p(x) = C(xe^x + e^x)$.

$y'_p(x) = Cxe^x + Ce^x = Ce^x(1+x)$.

~~$y''_p(x) = (1+x)Ce^x + Ce^x(1+x)^{-1}$~~.

$y''_p(x) =$ ~~$Ce^x(1+x) +$~~ $Ce^x[1+x+1)$

$y''_p(x) = Ce^x(2+x)$.

ubstitute into $y'' - y' = 2e^x$.

$Ce^x(2+x) - Cxe^x = 2e^x$

$C(2+x) - Cx = 2$.

$C[2+x-x] = 2$.

$C(2) = 2$.

$C = 1$.

$y_p(x) =$ ~~$2e^x$~~ xe^x.

$\therefore y(x) = y_c(x) + y_p(x)$.

$y(x) = Ae^x + Be^{-x} + xe^x$.

VARIATION OF PARAMETERS:

It is also known as the general method.

e.g. If $f(x) = \tan x$, $\tan x$ cannot be expressed in any of the form of undetermined coefficient.

Eg Consider $y'' + ay' + by = f(x)$,

If we have 2 linearly independent solution y_1 & y_2 of the homogeneous part, then we only know that

$y_c(x) = C_1 y_1(x) + C_2 y_2(x)$ which is the complementary function. We know that the solution of order 2 non-homogeneous equation appears in two form, which are $y_c(x)$ & $y_p(x)$.

If $y_p(x)$ is the P.I, then P.I must satisfy the property that $\frac{y_p(x)}{y_1(x)}$ and

$\frac{y_p(x)}{y_2(x)}$ are not constant.

Such that: $y_p(x) = C_1(x) y_1 + C_2(x) y_2$.

$\cos x \quad \frac{\cos^2 x}{\cos} \quad \frac{}{\cos x}$

$\cos x - \sec x.$

Where C_1 and C_2 in this case are not constant they are rather functions of x.

With $C_1(x)$ \$ $C_2(x)$ Satisfying the differential

$$C_1' = \frac{-f(x)\,y_2(x)}{y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x)}$$

$$C_2' = \frac{\ominus f(x)\,y_1(x)}{y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x)}.$$

Where $y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x)$ is the determinant of WRONSKIAN.

$$W = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} = Y_1Y_2' - Y_2Y_1' \neq 0.$$

∴ The general Solution will be

$$y(x) = y_c(x) + y_p(x).$$

Example: Solve the D-E

$$y'' + y = \tan x.$$

$m^2 + 1 = 0.$

$m^2 = -1$

$m = \pm i$

∴ $y_c(x) = A_1\cos x + A_2\sin x.$

$y_c(x) = A_0\cos x + B\sin x.$

$y_1(x) = \cos x, \quad y_2(x) = \sin x.$

Where A \$ B are arbitrary constant.

$y_p(x) = C_1(x)\,y_1 + C_2(x)\,y_2.$

$y_p(x) = C_1(x)\cos x + C_2(x)\sin x.$

$$\therefore C_1' = \frac{-\tan x \sin x}{\cos x \cdot \cos x + \sin x \sin x}$$

~~$C_1' = \frac{-\tan x \sin x}{\cos x[\cos x + \sin x]} =$~~

$$C_1' = -\tan x \sin x = -\frac{\sin x}{\cos x}.\sin x$$

$$C_1' = \frac{-\sin^2 x}{\cos x} = \cos x - \sec x.$$

$$C_1 = \int(\cos x - \sec x)\,dx = \sin x - \ln|\sec x + \tan x|.$$

$$\int \sec x\,dx = \int \frac{1}{\cos x}\,du.$$

Let $u = \cos x.$

$\frac{du}{dx} = -\sin x.$

$dx = \frac{-du}{\sin x}.$

$\int \frac{1}{u} \cdot \frac{-du}{\sin x}.$

$$C_2' = \frac{-\tan x \cos x.}{\cos x \cos x + \sin x \sin x} = -\tan x \cos x.$$

$$C_2' = \frac{-\sin x}{\cos x}.\cos x = -\sin x.$$

$$C_2 = \int -\sin x\,dx. = +\cos x$$

This implies that

$$y_p(x) = \sin x - \ln|\sec x + \tan x|\cos x + \cos x \sin x.$$

$$y_p(x) = \cos x\left[(\sin x - \ln|\sec x + \tan x|) + \sin x\right].$$

∴ The general Solution of the D.E is

$y(x) = A\cos x + B\sin x + (\sin x - \ln|\sec x + \tan x|)$
$\cos x + \cos x \sin x.$

③ Solve $y'' - 3y' + 2y = e^{5x}$.

Soln-

$y'' - 3y' + 2y = 0.$

$m^2 - 3m + 2 = 0$

$m_1 = 1, \; m_2 = 2$

$y_c(x) = Ae^x + Be^{2x}.$

$y_1(x) = e^x, \quad y_2(x) = e^{2x}.$

$y_p(x) = C_1(x)e^x + C_2(x)e^{2x}.$

$$C_1' = \frac{-e^{5x}e^{2x}}{e^x 2e^{2x} - e^{2x}\cdot e^x} = \frac{-e^{7x}}{2e^{3x} - e^{3x}}$$

$$C_1' = \frac{-e^{7x}}{e^{3x}} = -e^{4x}.$$

$$C_1 = \int -e^{4x}dx = \frac{-e^{4x}}{4}.$$

$$C_2' = \frac{-e^{5x}\cdot e^x}{e^x \cdot 2e^{2x} - e^{2x}e^x} = \frac{-e^{6x}}{e^{3x}}$$

$$C_2' = -e^{3x}$$

$$C_2 = \int -e^{3x}dx = \frac{-e^{3x}}{3}.$$

$$\therefore y_p(x) = \frac{-e^{4x}}{4}e^x + \left(\frac{-e^{3x}}{3}\right)e^{2x}.$$

$$y_p(x) = \frac{-e^{5x}}{4} - \frac{e^{5x}}{3}$$

$$y_p(x) = \frac{-3e^{5x} - 4e^{5x}}{12}$$

$$y_p(x) = \frac{\cancel{-7}e^{5x}}{12} \quad ✗ \quad y_p = \frac{e^{5x}}{12}.$$

∴ The general solution will be

$$y(x) = Ae^x + Be^{2x} - \frac{\overset{\times}{7}}{12}e^{5x}.$$
$$= Ae^x + Be^{2x} + \frac{e^{5x}}{12}.$$

Exercise:

① $y'' + 4y = 8\sin x$

② $y'' + 9y = \sec 3x.$

Solution:

$y'' + 4y = 8\sin x.$

Let $f(x) = 0$, which in this case is $8\sin x$, to make the given equation homogeneous,

$\therefore y'' + 4y = 0.$

The characteristics equation is

$m^2 + 4 = 0.$

$\therefore m^2 = -4$

$m = \pm\sqrt{-4} = \pm 2i$

The complementary solution is

$y_c = (A\cos 2x + B\sin 2x).$

For the integral part, let

$y_p(x) = C\sin x.$

$\therefore y_p'(x) = +C\cos x.$

$y''_p(x) = -c\sin x$

$\therefore -c\sin x + 4(c\sin x) = 8\sin x$

$-c\sin x + 4c\sin x = 8\sin x$

$3c\sin x = 8\sin x$

$3c = 8$

$c = 8/3$

$\therefore y_p(x) = \frac{8}{3}\sin x$

$\therefore$ The general Solution of $y'' + 4y = 8\sin x$ is

$y(x) = y_p(x) + y_c(x)$

$= (A\cos 2x + B\sin 2x) + \frac{8}{3}\sin x$

(ii) OR Using Variation of Parameters Method, Knowing fully well that the complementary part is $y_c(x) = A\cos 2x + B\sin 2x$.

For P.I, $y_p(x) = C_1(x)y_1~~(x)~~ + C_2(x)y_2$.

$y_1 = \cos 2x, \quad y_2 = \sin 2x$.

But $C_1' = \dfrac{-f(x)y_2(x)}{y_1(x)y_2'(x) - y_2(x)y_1'(x)}$.

$C_1' = \dfrac{-8\sin x(\sin 2x)}{\cos 2x(2\cos 2x) - \sin 2x(-2\sin 2x)}$.

$C_1' = \dfrac{-8\sin x(\sin 2x)}{2\cos^2 2x + 2\sin^2 2x}$

$C_1' = \dfrac{-8\sin x(\sin 2x)}{2(\cos^2 2x + \sin^2 2x)} =$

② $y'' + 9y = \sec 3x$.

$m^2 + 9 = 0$

$m^2 = -9$

$m = \pm 3i$

$y_c(x) = A\cos 3x + B\sin 3x$.

P.I, $y_p(x) = C_1(x)y_1 + C_2(x)y_2$.

$C_1' = \dfrac{-f(x)y_2}{y_1y_2' - y_2y_1'} = \dfrac{-\sec 3x(\sin 3x)}{\cos 3x(3\cos 3x) - \sin 3x(-3\sin 3x)}$

$C_1' = \dfrac{-\sec 3x\sin 3x}{3\cos^2 3x + 3\sin^2 3x} = \dfrac{-\sec 3x\sin 3x}{3(\cos^2 3x + \sin^2 3x)}$.

$C_1' = \dfrac{-\sec 3x\sin 3x}{3}$.

$C_1 = \frac{-1}{3}\int \sec 3x\sin 3x\,dx = \frac{-1}{3}\int \frac{\sin 3x}{\cos 3x}dx$.

Let $u = \cos 3x$.

$\frac{du}{dx} = -3\sin 3x$.

$dx = \dfrac{-du}{3\sin 3x}$.

$C_1 = \frac{-1}{3}\int \frac{\sin 3x}{u} \cdot \frac{-du}{3\sin 3x} = \frac{1}{9}\int \frac{du}{u}$.

$C_1 = \frac{1}{9}\ln|\cos 3x|$

$C_2' = ~~8\sin x~~ \dfrac{-\sec 3x\cos 3x}{3(\cos^2 3x + \sin^2 3x)}$.

$C_2' = \dfrac{-\sec 3x\cos 3x}{3} =$

$C_2 = \frac{-1}{3}\int \frac{1}{\cos 3x} \cdot \cos 3x\,dx = \frac{-1}{3}\int dx$

$C_2 = \frac{-1}{3}x$.

$$\therefore y_p(x) = \frac{1}{9}\ln|\cos 3x|(\cos 3x) + \left(\frac{-x}{3} \cdot \sin 3x\right).$$

$$= \frac{1}{9}\cos 3x\left(\ln|\cos 3x|\right) - \frac{x\sin 3x}{3}.$$

$$= \frac{1}{9}\left[(\cos 3x)\ln|\cos 3x| - 3x\sin 3x\right].$$

$\therefore$ The general Solution is

$$y(x) = A\cos 3x + B\sin 3x + \frac{1}{9}\left[(\cos 3x)\ln|\cos 3x| - 3x(\sin 3x)\right].$$

Exercises.

* Solve the differential equations

① $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$

This is a non-linear equation of the Case where x is missing and it follows the general form $G(y, y', y'') = 0.$

So, Let $v = \frac{dy}{dx} = y'$

$$\therefore \frac{d^2y}{dx^2} = v' = \frac{dv}{dx}.$$

$$\frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = v\frac{dv}{dy}.$$

$$\therefore yv\frac{dv}{dy} + v^2 = 0.$$

$$-yv\frac{dv}{dy} = -v^2.$$

$$y\frac{dv}{dy} = \frac{-v^2}{v} = -v.$$

$$ydv = -vdy.$$

$$\frac{y}{dy} = \frac{-v}{dv}$$

$$\int \frac{dy}{y} = \int \frac{-dv}{v}$$

$$\ln y = -\ln v + \ln c.$$

$$\ln y = \ln v^{-1} + \ln c$$

$$\ln y = \ln cv^{-1}$$

$$y = cv^{-1}.$$

error.

$$yv\frac{dv}{dy} = -v^2$$

$$y\frac{dv}{dy} = -v.$$

$$ydv = -vdy.$$

$$\int \frac{dv}{v} = -\int \frac{dy}{y}$$

$$\ln v = -\ln y + \ln c.$$

$$\ln v = \ln y^{-1} + \ln c.$$

$$\ln v = \ln cy^{-1}$$

$$v = cy^{-1} \qquad v \neq \frac{c}{y}$$

$$\frac{dy}{dx} = cy^{-1}. \qquad vy = c$$

$$\frac{dy}{dx}y = c.$$

$$\int \frac{dy}{cy^{-1}} = \int dx. \qquad \int ydy = \int cdx.$$

$$\frac{y^2}{2} = cx + k.$$

$$y^2 = 2(cx + k).$$

$$y = \sqrt{2(cx+k)} = \left[2(cx+k)\right]^{1/2}.$$

② $2x\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 - 1.$

$\frac{1}{\sqrt{\cdot}}$ $(v-1)(v+1)$ $\frac{1}{x^2-a^2}$

v^2-1 $\frac{1}{a}$

Soln.

• This is also a second order non-linear equation of the Case where y is missing,

$\therefore G(x, y', y'') = 0$

Let $v = \frac{dy}{dx} = y'$.

$\frac{d^2y}{dx^2} = v' = y''$

$\therefore G(x, v, v') = 0$.

$\therefore 2xv' = v^2-1$.

$2x\frac{dv}{dx} = v^2-1$

$\frac{2x\,dv}{v^2-1} = dx$.

$\int\frac{dv}{v^2-1} = \int\frac{dx}{2x}$

$\int\frac{1}{(v+1)(v-1)}dv = \frac{1}{2}\ln x + c$.

$\frac{1}{(v+1)(v-1)} = \frac{A}{v+1} + \frac{B}{v-1}$.

$1 = A(v-1) + B(v+1)$.

when $v=1$. when $v=-1$.

$1 = 2B$ $1 = -2A$.

$B = 1/2$. $A = -1/2$.

$\therefore \frac{1}{(v+1)(v-1)} = \frac{1}{2(v-1)} - \frac{1}{2(v+1)}$.

$\therefore \int\frac{1}{2(v-1)}dv - \int\frac{1}{2(v+1)}dv = \frac{1}{2}\ln x + c$.

$\frac{1}{2}\ln(v-1) - \frac{1}{2}\ln(v+1) \pm \frac{1}{2}\ln x + \ln C$.

$\ln(v-1)^{1/2} - \ln(v+1)^{1/2} = \ln x^{1/2} + \ln C$.

$\ln\left(\frac{(v-1)^{1/2}}{(v+1)^{1/2}}\right) = \ln Cx^{1/2}$.

$\left(\sqrt{\frac{v+1}{v+1}}\right)^2 = (C\sqrt{x})^2$

$\frac{v-1}{v+1} = Ax$.

$v-1 = Ax(v+1)$.

$v-1 = Axv + Ax$.

$v - Axv = Ax+1$.

$v(1-Ax) = Ax+1$.

$v = \frac{Ax+1}{1-Ax}$.

$\frac{dy}{dx} = \frac{Ax+1}{1-Ax}$.

$\int dy = \int\frac{Ax+1}{1-Ax}dx$.

$y = \int\frac{Ax+1}{1-Ax}dx$.
*

* $\int\left(1 + \frac{2}{1-Ax}\right)dx$

$\int dx + 2\int\frac{1}{1-Ax}dx = x + 2\frac{\ln(1-Ax)}{-A}$.

$$\boxed{y = x - \frac{2}{A}\ln(1-Ax) + C}$$

Let $u = 1-Ax$

$\frac{du}{dx} = -A$; $dx = \frac{-du}{A}$

$u-1 = -Ax$.

$1-u = Ax$.

$x = \frac{1-u}{A}$

$\int\frac{1-u+1}{u}\cdot\frac{-du}{A}$.

$\int\frac{2-u}{u}\cdot\frac{-du}{A}$.

$-\frac{1}{A}\int\frac{2-u}{u}du$.

$-\frac{1}{A}\left[\int\frac{2}{u} - 1\,du\right]$.

$\frac{-1}{A}\cdot[2\ln u - u]$.

$-\frac{2}{A}\ln(1-Ax) + \frac{1-Ax}{A}$.

$-\frac{2}{A}[\ln(1-Ax) - 2$

$1-Ax\overline{)\,Ax+1}$

$1-Ax$

$+\frac{-Ax+1}{2}$

$A\sqrt{1-Ax}$

$A\sqrt{\frac{Ax+1}{A}}$

✗. Exercise.

① Solve $y'' - 5y' + 6 = 0$

(ii) $y'' + cy = 0$ if $c > 0$.

$m^2 + c = 0$.

$m^2 = -c$

$m = \pm\sqrt{-c}$

$m = \pm ic$ $\alpha = 0,\ \beta = c$.

$\therefore y(x) = [A\cos cx + B\sin cx]$.

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HIGHER ORDER DERIVATE.

(i) $Y''' + 4y'' + Y' - 6y = 0$.

This equation is an example of Order 3 homogeneous Linear equation because $f(x) = 0$. So, we need to find the auxillary equation of this DE

$m^3 + 4m^2 + m - 6 = 0$

$1\ 4\ 1\ -6$

When solved, we have

$m_1 = 1,\ m_2 = -2,\ m_3 = -3$

$y_1 = A_1e^x,\ y_2 = A_2e^{-2x},\ y_3 = A_3e^{-3x}$.

But $Y(x) = y_1(x) + y_2(x) + y_3(x)$

$\therefore y(x) = A_1e^x + A_2e^{-2x} + A_3e^{-4x}$.

② $Y^{(IV)} + 16y''' + 96y'' + 256y' + 256y = 0$.

This is order 4 homogeneous linear equation, the process of solving it is the same as the ones above.

The auxillary equation of this DE is

$m^4 + 16m^3 + 96m^2 + 256m + 256 = 0$

$m_1 = -4,\ m_2 = -4,\ m_3 = -4,\ m_4 = -4$.

Therefore the general solution is

$y_1(x) = A_1e^{-4x},\ y_2(x) = A_2xe^{-4x}$

$y_3(x) = A_3x^2e^{-4x},\ y_4(x) = A_4x^3e^{-4x}$.

$\therefore y(x) = A_1e^{-4x} + A_2xe^{-4x} + A_3x^2e^{-4x} + A_4x^3e^{4x}$.

NON HOMOGENEOUS TYPE.

In this case, the best method to use is the Variation of Parameter.

If $f(x) \neq 0$ we have non-homogeneous equation.

$P.I = Y_p(x) = C_1(x)Y_1(x) + C_2(x)Y_2(x) + \ldots + C_n(x)Y_n(x)$.

$$Y_p'(x) = C_1y_1' + C_2Y_2' + \ldots + C_ny_n' + \underbrace{y_1C_1' + C_2'Y_2 + \ldots + C_n'y_n}_{0} \text{ --- a.}$$

Let $C_1'y_1 + C_2'y_2 + \ldots + C_n'y_n = 0$.

$\therefore Y_p'(x) = C_1y_1' + C_2y_2' + \ldots + C_ny_n'$.

$y_p'' = C_1y_1'' + C_2y_2'' + \ldots + C_ny_n'' + C_1'y_1' + C_2'y_2' + \ldots C_n'y_n' = 0$ --- (b)

$\therefore y_p'' = C_1y_1'' + C_2y_2'' + \ldots + C_ny_n''$

$$\therefore y'''_p = C_1y_1''' + C_2y_2''' + \cdots + C_ny_n''' + \underbrace{C_1'y_1'' + C_2'y_2'' + \cdots + C_n'y_n''}_{=0} \cdots c.$$

$$\therefore y^{(n)}_p = C_1y^n + C_2y_n^n + \cdots + C_ny_n^n + \underbrace{C_1'y_1^{n-1} + C_2'y_2^{n-1} + \cdots + C_n'y_n^{n-1}}_{f(x)} \cdots d$$

$$\therefore y_p^n = C_1y_1^n + C_2y_n^n + \cdots + C_ny_n^n + f(x).$$

Solve a, b, c and d Simultaneously, i.e

$$C_1'y_1 + C_2'y_2 + \cdots + C_n'y_n = 0$$
$$C_1'y_1' + C_2'y_2' + \cdots + C_n'y_n' = 0$$
$$C_1'y_1'' + C_2'y_2'' + \cdots + C_n'y_n'' = 0$$
$$\vdots$$
$$C_1'y_1^{n-1} + C_2'y_2^{n-1} + \cdots + C_n'y_n^{n-1} = f(x)$$

Solving this using cramer's rule,

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{n-1} & y_2^{n-1} & & y_n^{n-1} \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \\ \vdots \\ C_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(x) \end{bmatrix}$$

$$C_1' = \frac{W_1}{W} \qquad \text{where, } W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \end{vmatrix}$$

$$C_2' = \frac{W_2}{W}$$
$$\vdots$$
$$C_n' = \frac{W_n}{W}.$$

This equation stated is already a Cramer's.

$$\therefore W_1 = \begin{vmatrix} 0 & y_2 & y_n \\ 0 & y_2' & y_n' \\ \vdots & \vdots & \vdots \\ f(x) & y_2^{n-1} \cdots & y_n^{n-1} \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_n \\ y_1' & 0 & y_n' \\ \vdots & \vdots & \vdots \\ y_1^{n-1} & f(x) & y_n^{n-1} \end{vmatrix}$$

For W_n

$$W_n = \begin{vmatrix} y_1 & y_2 & \cdots & 0 \\ y_1' & y_2' & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ y_1^{n-1} & y_2^{n-1} & \cdots & f(x) \end{vmatrix}.$$

08-06-18.

For 3rd Order,

$$y_p(x) = C_1(x)y_1(x) + C_2(x)y_2(x) + C_3(x)y_3(x).$$

$$y_p'(x) = (C_1y_1' + C_2y_2' + C_3y_3') + \underbrace{(C_1'y_1 + C_2'y_2 + C_3'y_3)}_{0} \cdots (a).$$

$$y_p'' = (C_1y_1'' + C_2y_2'' + C_3y_3'') + \underbrace{(C_1' + y_1' + C_2'y_2' + C_3'y_3')}_{0} \cdots (b)$$

$$y_p''' = (C_1y_1''' + C_2y_2''' + C_3y_3''') + \underbrace{(C_1'y_1'' + C_2'y_2'' + C_3'y_3'')}_{f(x)} \cdots (c)$$

Solve a, b & c Simultaneously,

$$C_1y_1 + C_2y_2 + C_3y_3 = 0$$
$$C_1y_1' + C_2y_2' + C_3y_3' = 0$$
$$C_1y_1'' + C_2y_2'' + C_3y_3'' = f(x).$$

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$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}\begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f(x) \end{bmatrix}.$$

Using Cramer's rule,

$$c_1' = \frac{W_1}{W}, \quad c_2' = \frac{W_2}{W}, \quad c_3' = \frac{W_3}{W}.$$

Where

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0.$$

W → WRONSKIAN.

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}.$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}.$$

Exercise!

Solve $y''' + 3y' - 2y = 3e^{-x}$.

$y_1 = A_1 e^{-x}$, $y_2 = A_2 x e^{-x}$, $y_3 = A_3 e^{2x}$.

Soln.

$y''' - 3y' - 2y = 0$.

$m^3 - 3m - 2 = 0$.

$m_1 = -1, \quad m_2 = -1, \quad m_3 = 2$.

$y_c(x) = A_1 e^{-x} + A_2 x e^{-x} + A_3 e^{2x}$.

$y_p(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{2x}$.

$y_p' = -c_1 e^{-x} - c_2 x e^{-x} + c_2 e^{-x} + 2c_3 e^{2x}$

$= -c_1 e^{-x} + c_2 (e^{-x} - x e^{-x}) + 2c_3 e^{2x} +$

$\underbrace{(c_1' e^{-x} + c_2' x e^{-x} + c_3' e^{2x})}_{=0}$ ---- ⓐ.

$y_p'' = +c_1 e^{-x} + c_2 [-e^{-x} + x e^{-x} - e^{-x}] + 4c_3 e^{2x} +$

$\underbrace{[-c_1' e^{-x} + c_2' [e^{-x} - x e^{-x}] + 2c_3' e^{2x}]}_{=0}$ ---- (b).

$y_p''' = [-c_1 e^{-x} + c_2 [2e^{-x} + e^{-x} - x e^{-x}] + 8c_3 e^{2x}] +$

$\underbrace{[c_1' e^{-x} + c_2' [2e^{-x} + x e^{-x}] + 4c_3' e^{2x}]}$ ---- (c)

$f(x) = 3e^{-x}$

Combining equations (a, b & c) and solve

Simultaneously, we get.

$c_1' e^{-x} + c_2' x e^{-x} + c_3' e^{2x} = 0$.

$-c_1' e^{-x} + c_2' (1-x) e^{-x} + 2c_3' e^{2x} = 0$.

$c_1' e^{-x} + c_2' (x-2) e^{-x} + 4c_3' e^{2x} = 3e^{-x}$.

$$\begin{bmatrix} e^{-x} & x e^{-x} & e^{2x} \\ -e^{-x} & (1-x) e^{-x} & 2e^{2x} \\ e^{-x} & (x-2) e^{-x} & 4e^{2x} \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3e^{-x} \end{bmatrix}.$$

Using crammer's Rule, we have.

$C_1' = \frac{W_1}{W}$, $C_2' = \frac{W_2}{W}$, $C_3' = \frac{W_3}{W}$.

$$W = \begin{vmatrix} e^{-x} & xe^{-x} & e^{2x} \\ -e^{-x} & (1-x)e^{-x} & 2e^{2x} \\ e^{-x} & (x-2)e^{-x} & 4e^{2x} \end{vmatrix} =$$

$$e^{-x} \begin{vmatrix} (1-x)e^{-x} & 2e^{2x} \\ (x-2)e^{-x} & 4e^{2x} \end{vmatrix} - xe^{-x} \begin{vmatrix} -e^{-x} & 2e^{2x} \\ e^{-x} & 4e^{2x} \end{vmatrix} + e^{2x} \begin{vmatrix} -e^{-x} & (1-x)e^{-x} \\ e^{-x} & (x-2)e^{-x} \end{vmatrix}.$$

$$= e^{-x}\left[4e^{2x}(1-x)e^{-x} - (x-2)e^{-x}\cdot 2e^{2x}\right] -$$

$$xe^{-x}\left[4e^{2x}(-e^{-x}) - e^{-x}(2e^{2x})\right] +$$

$$e^{2x}\left[-e^{-x}(x-2)e^{-x} - e^{-x}(1-x)e^{-x}\right].$$

$$= -(-4x - 2x) + (4 - 4x - (2x-4)] + (-x+2) - 1 + x)$$

$$= 4x + 2x + (8 - 6x) + 1 = 6x + 8 - 6x + 1$$

$$= 9.$$

$$W_1 = \begin{vmatrix} 0 & xe^{-x} & e^{2x} \\ 0 & (1-x)e^{-x} & 2e^{2x} \\ 3e^{-x} & (x-2)e^{-x} & 4e^{2x} \end{vmatrix}.$$

$$= -xe^{-x}\left[-3e^{-x}\cdot 2e^{2x}\right] + e^{2x}\left[-3e^{-x}(1-x)e^{-x}\right]$$

$$= -xe^{-x}(-6e^{x}) + e^{2x}(-3(1-x)e^{-2x}).$$

$$= 6x - 3 + 3x = 9x - 3.$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 & e^{2x} \\ -e^{-x} & 0 & 2e^{2x} \\ e^{-x} & 3e^{-x} & 4e^{2x} \end{vmatrix}$$

$$= e^{-x}\left[-3e^{-x}(2e^{2x})\right] + e^{2x}\left[-3e^{-2x}\right]$$

$$= -6 - 3 = -9.$$

$$W_3 = \begin{vmatrix} e^{-x} & xe^{-x} & 0 \\ -e^{-x} & (1-x)e^{-x} & 0 \\ e^{-x} & (x-2)e^{-x} & 3e^{-x} \end{vmatrix}$$

$$= e^{-x}\left(3(1-x)e^{-2x}\right) - xe^{-x}\left(-(x-2)e^{-2x} - (1-x)e^{-2x}\right)$$

$$= (3-3x)e^{-3x} - xe^{-x}\left((2-x)e^{-2x} - ((1-x)e^{-2x}\right).$$

$$= 3e^{-3x} - 3xe^{-3x} - (2xe^{-x} + x^2e^{-x})e^{-2x} + (1-x)xe^{-3x}).$$

$$= 3e^{-3x} - 3xe^{-3x} + 2xe^{-3x} \cancel{+ x^2e^{-3x}} + xe^{-3x} - \cancel{x^2e^{-3x}}$$

$$= 3e^{3x}.$$

$$\therefore C_1' = \frac{9x-3}{9} = x - \frac{1}{3}. \qquad C_1 = \int\left(x - \frac{1}{3}\right)dx = \frac{x^2}{2} - \frac{1x}{3}$$

$$C_2' = \frac{-9}{9} = -1, \qquad C_2 = \int -1\,dx = -x$$

$$C_3' = \frac{3e^{3x}}{9} = \frac{1}{3}e^{3x} \qquad C_3 = \frac{1}{3}\int e^{3x}dx = \frac{-1}{9}e^{3x}.$$

$$\therefore y_p(x) = \left(\frac{x^2}{2} - \frac{1}{3}x\right)e^{-x} + x^2e^{-x} - \frac{e^{3x}}{9}(e^{2x}).$$

$$= \frac{x^2}{2}e^{-x} - \frac{1}{3}xe^{-x} - x^2e^{-x} - \frac{e^{-x}}{9}.$$

$$= e^{-x}\left(\frac{x^2}{2} - \frac{x}{3} - x^2 - \frac{1}{9}\right).$$

$$y(x) = y_c + y_p$$

$$= A_1e^{-x} + A_2xe^{-x} + A_3e^{2x} + e^{-x}\left[\frac{x^2}{2} - \frac{x}{3} - x^2 - \frac{1}{9}\right].$$

0·33 $\frac{33}{100}$ $\sqrt{}$

$$100\overline{)300}\ 0{\cdot}33$$
-300

$120+1($

14-06-18. ERROR AND INTERPOLATION.

TYPES OF ERROR.

① GROSS ERROR OR BLUNDER: This is an error committed when a wrong answer is written in place of a correct answer. e.g 0·5791 instead of 0·5971.

② TRUNCATION ERROR: Is the error committed when an ~~infinite~~ process is replaced by a finite process. e.g

$$(1+x)^n = 1 + {}^nC_1 x + \frac{n(n-1)}{2!}x^2 + ----$$

$$= 1 + nx + {}^nC_2 x^2 + ----.$$

③ ROUND-OFF ERROR: This is the error committed when a certain arithmetic calculation is obtained as fraction but we decide to express it as decimal.

e.g $\frac{1}{3}, \frac{2}{3}, \frac{6}{7}$ etc.

ERROR PROPAGATION.

Let X be an exact value

Let X^* be an approximated value.

$$E = X - X^*$$

$$E = X^* - X.$$

⇒ Absolute Error $= |X - X^*|$.

$$\text{Relative Error} = \frac{\text{Actual Error}}{\text{Actual value.}}$$

$$RE = \frac{|X - X^*|}{X} = \frac{\Delta X}{X}.$$

If X is a real number or a real value which in general has an infinite decimal representation, then we say that X has been rounded-off to d-decimal places if this inequality is correct.

$$|E^d| = |X - X^d| \leq \frac{1}{2} \times 10^{-d}.$$

E.g If $X = 0{\cdot}6667$, $X^* = 0{\cdot}67 = X^2$

$$E^2 = |0{\cdot}6667 - 0{\cdot}67| \leq \frac{1}{2} \times 10^{-2}.$$

$$= 0{\cdot}0033 \leq 0{\cdot}005.$$

Since 0·0033 is actually less than 0·005, hence, the approximation from 0·6667 to 0·67 is a good one.

② Let $X = 0{\cdot}333333$, $X' = 0{\cdot}3$

$$E = |0{\cdot}333333 - 0{\cdot}3| \leq \frac{1}{2} \times 10^{-1}$$

$$= 0{\cdot}033333 \leq 0{\cdot}05.$$

The approximation is ~~not~~ a good one because the inequality is correct.

$E^d = |x - x^d| \le \frac{1}{2}\times 10^{-d}$.   0.047

③ $X = 0.5555$ $X' = 0.5$

$E = |0.5555 - 0.5| \le \frac{1}{2}\times 10^{-1}$

$= 0.0555 \le 0.05$.

Since the inequality is not true, hence the approximation is not a good one.

⇒ Let X and Y represent two real values and X^* and Y^* respresent their respective approximation and E_x and E_y representing error in X and error in Y, then /

$X^* = X - E_x$, $Y^* = Y - E_y$

$\Rightarrow E_x = X - X^*$, $\Rightarrow E_y = Y - Y^*$.

① ADDITION:

$X^* + Y^* = (X - E_x) + (Y - E_y)$.

$\Rightarrow (X+Y) - (E_x + E_y)$.

$\therefore X^* + Y^* = (X+Y) - (E_x + E_y)$.

E.g $X = 4.701$, $X^* = 4.7$

$Y = 2.346$, $Y^* = 2.4$

$4.7 + 2.4 = (4.701 + 2.346) - (E_x + E_y)$

$E_x = |X - X^*| = 0.001$

$E_y = |Y - Y^*| = 0.054$

$\therefore 7.1 = (7.047) - (0.001 + 0.054)$

$7.1 = 7.047 - 0.055$

$7.1 = 6.992$.

If $Y^* = 2.3$, $E_y = 2.346 - 2.3 = 0.046$

$\therefore X^* + Y^* = 7.047 - 0.047$

$7 = 7$

SUBTRACTION:

$X^* - Y^* = (X - E_x) - (Y - E_y)$.

$= (X - Y) + (E_y - E_x)$.

21-06-18.

MULTIPLICATION:

$X^* \times Y^* = (X - E_x)(Y - E_y)$.

$= XY - XE_y - YE_x + E_xE_y$

$= XY - XE_y - YE_x$

Where $E_xE_y \approx 0$.

$\therefore X^*Y^* - XY = -[XE_y + YE_x]$.

$XY - X^*Y^* = XE_y + YE_x$.

But Relative Error = $\frac{\text{Actual Error}}{\text{True value}}$

$Rel_{error} = \frac{XY - X^*Y^*}{XY} = \frac{XE_y + YE_x}{X.Y}$.

$= \frac{XE_y}{XY} + \frac{YE_x}{XY} = \frac{E_y}{Y} + \frac{E_x}{X}$.

∴ The relative error in the product equals the sum of the relative error in the no we are multiplying.

DIVISION:

$\frac{X^*}{Y^*} = \frac{X - E_x}{Y - E_y}$

$\frac{2}{3} \div \frac{3}{7}$

$X \quad Y$

$\frac{2}{3} \times \frac{7}{3}$

$$\frac{X}{Y} - \frac{X^*}{Y^*} = \left(\frac{X}{Y}\right) - \left(\frac{X-E_x}{Y-E_y}\right)$$

$$= \frac{X(Y-E_y) - Y(X-E_x)}{Y(Y-E_y)}$$

$$= \frac{XY - XE_y - XY + YE_x}{Y(Y-E_y)} = \frac{YE_x - XE_y}{Y(Y-E_y)}$$

~~Relative Error = X/Y~~

$$\therefore \frac{X}{Y} - \frac{X^*}{Y^*} = \frac{YE_x - XE_y}{Y(Y-E_y)}$$

$$\text{Relative Error} = \frac{\frac{YE_x - XE_y}{Y(Y-E_y)}}{X/Y}$$

$$= \frac{YE_x - XE_y}{Y(Y-E_y)} \times \frac{Y}{X}$$

$$= \frac{YE_x - XE_y}{X(Y-E_y)}$$

$$= \frac{1}{Y(Y-E_y)} \times YE_x - XE_y \times \frac{Y}{X}$$

$$= \frac{1}{Y(Y-E_y)} \times \frac{Y^2E_x - XYE_y}{X}$$

$$= \frac{1}{Y^2 - YE_y} \times \left(\frac{Y^2E_x}{X} - \frac{Y\cancel{X}E_y}{\cancel{X}}\right)$$

$$= \frac{Y^2E_x}{X(Y^2-YE_y)} - \frac{YE_y}{Y^2-YE_y}$$

Linear, Quadratic, Lagrange.

Linear → Two points are needed

Quadratic → 3 points are needed.

Lagrange → more than 3 points are needed.

Exercise: Three numbers X, Y and Z when rounded up to 3 d.p gives the following values 4.701, 0.832, and 2.413 respectively. Evaluate an approximation of a product ~~X, Y and Z~~ XYZ and the determine the error involved.

Soln

INTERPOLATION

Given n+1 data with (x_i, f_i) where i takes values from 0, 1, ---- n, we seek a polynomial $P_n(x)$ which takes f_i at every x_i. The $P_n(x) = f_i(x_i)$

The polynomial $P_n(x)$ is called interpolating polynomial and f_i represent starting mathematical function. The process of obtaining $P_n(x_i)$ is called INTERPOLATION.

Given $x_1, x_2, \ldots x_{n+1}$,

$P_n(x_n)$ = Is Interpolated.

But $P_n(x_{n+3})$ = Extrapolation.

$P_n(x_0)$ = Intrapolation.

29-06-18. TYPES OF INTERPOLATION.

(i) Linear Interpolation → Linear equation.

(ii) Quadratic Interpolation.

(iii) Lagrange Interpolation.

(i) Linear : Minimum of Two points are needed $y = mx + c$.

(ii) Quadratic : Minimum of three points are needed.

(iii) Lagrange : More than three points.

LINEAR INTERPOLATION.

⇒ This is the interpolation by means of straight line through two points (x_0, f_0) and (x_1, f_1) given by $P_1(x) = f_0 + (x - x_0)\, f(x_0, x_1)$ → order 1

Where $f(x_0, x_1) = \dfrac{f_1 - f_0}{x_1 - x_0}$.

Polynomial Gradient of interpolation.

If $x = x_0$, $P_1(x) = f_0$.

It can also be written in another form;

$$P_1(x) = f_0 + (x - x_0)\, \frac{f_1 - f_0}{x_1 - x_0}$$

If $x = x_1$, $P_1(x) = f_1$

Example:

Estimate the population of Nigeria in 1999 given that;

Year	1990	2006
Population (m)	120	150.

Soln.

$X_0 = 1990$, $x_1 = 2006$.

$(x - x_0)(x - x_1)\, f(x_0, x_1, x_2)$

$f_0 = 120$, $f_1 = 150$.

$x = 1999$, $f_{(x)} = ?$

$$P_1(x) = f_0 + (x - x_0) \cdot \frac{f_1 - f_0}{x_1 - x_0}.$$

$$= 120 + (1999 - 1990) \cdot \frac{150 - 120}{2006 - 1990}.$$

$$P_1(x) = 120 + 9 \cdot \left(\frac{30}{16}\right).$$

$\therefore P_1(x) = 136.875m$

QUADRATIC INTERPOLATION.

This is interpolation of Second order polynomial with a curve through the points (x_0, f_0), (x_1, f_1); (x_2, f_2) given by

$P_2(x) = f_0 + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1$

Where $f(x_0, x_1, x_2) = \dfrac{f(x_1, x_2) - f(x_0 x_1)}{x_2 - x_0}$.

$f(x_1 x_2) = \dfrac{f_2 - f_1}{x_2 - x_1}$, $f(x_0 x_1) = \dfrac{f_1 - f_0}{x_1 - x_0}$.

$$\therefore f(x_0 x_1 x_2) = \frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}$$

$$P_2(x) = f_0 + (x - x_0)\left(\frac{f_1 - f_0}{x_1 - x_0}\right) + (x - x_0)(x - x_1)\left[\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}\right].$$

Example:

Compute the interpolation $f(0{\cdot}9)$ given that $f(0{\cdot}5) = 0{\cdot}479$, $f(1{\cdot}0) = 0{\cdot}841$, $f(2{\cdot}0) = 0{\cdot}909$.

Soln.

$x_0 = 0{\cdot}5$, $x_1 = 1{\cdot}0$, $x_2 = 2{\cdot}0$

$x = 0{\cdot}9$.

$f_0 = 0{\cdot}479$, $f_1 = 0{\cdot}841$, $f_2 = 0{\cdot}909$.

$f_x = ?$

$$P_2(x) = f_0 + (x-x_0)\left(\frac{f_1 - f_0}{x - x_0}\right) + (x-x_0)(x-x_1)$$

$$f\left[x_0 x_1 x_2\right].$$

$$P_2(x) = 0{\cdot}479 + (0{\cdot}9 - 0{\cdot}5)\left(\frac{0{\cdot}841 - 0{\cdot}479}{1 - 0{\cdot}5}\right) +$$

$$(0{\cdot}9 - 0{\cdot}5)(0{\cdot}9 - 1{\cdot}0)\left[\frac{0{\cdot}909 - 0{\cdot}841}{2{\cdot}0 - 1{\cdot}0} - \frac{0{\cdot}841 - 0{\cdot}479}{1{\cdot}0 - 0{\cdot}5}\right]$$

$= 0{\cdot}8045$.

LAGRANGE INTERPOLATION.

This involves interpolation of polynomial with order $n > 2$. Given a table of nth values. $(x_0, f_0), (x_1, f_1), \ldots (x_n, f_n)$, we require to evaluate the value of $P_n(x) = L_n(x)$

$$= \sum_{K=0}^{n} \frac{L_K(x)}{L_K(x_K)} f(x_K).$$

$$\frac{L_K(x)}{L_K(x_K)} = \frac{(x-x_0)(x-x_1)\ldots(x-x_j)(x-x_{K-1})}{(x_j-x_0)(x_j-x_1)\ldots(x_j-x_{j-1})\ldots(x_j-x_n)}.$$

$$L_3(x) = \frac{L_3(x)}{L_3(x_3)} = \frac{L_3(x)}{L_3(x_3)}.$$

$$= \frac{L_0(x)}{L_0(x_0)} + \frac{L_1(x)}{L_1(x_1)} + \frac{L_2(x)}{L_2(x_2)} + \frac{L_3(x)}{L_3(x_3)}.$$

$$P_3(x) = \frac{L_0(x)}{L_0(x_0)} f_0 + \frac{L_1(x)}{L_1(x_1)} f_1 + \frac{L_2(x)}{L_2(x_0)} f_2 + \frac{L_3 x}{L_3(x_3)} f_3.$$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f_3$$

Example: Consider the following,

x	0	2	3	5
$f(x)$	1	3	2	5

Find the Lagrange interpolation polynomial for the values.

Soln.

$x_0 = 0$, $x_1 = 2$, $x_2 = 3$, $x_3 = 5$

2.578848.

$f_0=1,\ f_1=3,\ f_2=2,\ f_3=5.$

$P_3(x) = \frac{3x^3}{10} - \frac{15}{6}x^2 + \frac{62}{15}x + 1.$

② Let $f(x) = \ln x$. Estimate the value of $\ln(0.6)$ from the table

x	0.4	0.5	0.7	0.8
fx	-0.9165	-0.6931	-0.3567	-0.2231.

Soln.

$x=0.6,\ x_0=0.4,\ x_1=0.5,\ x_2=0.7$

$x_3=0.8$

$f_0=-0.9165,\ f_1=-0.6931,\ f_2=-0.3567$

$f_3=-0.2231.$

$P_3(x) = -0.5100.$

③ Find ln 9.2 using the table below:

x	9.0	9.5	10.0	11.0
$\ln x$	2.69722	2.25129	2.30259	2.39790.

~~$x = 9.2$~~, Soln.

$x=9.2,\ x_0=9.0,\ x_1=9.5,\ x_2=10.0$

$x_3=11.0$

$f_0=2.69722,\ f_1=2.25129,\ f_2=2.30289$

$f_3=2.39790,\ f_x=?$

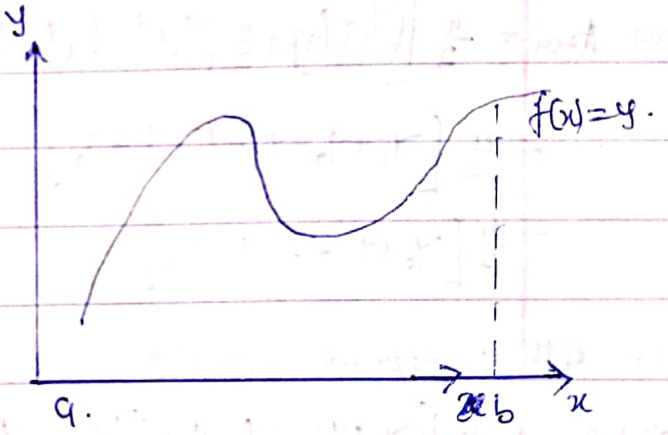
$P_3(x) = 2.21920$

NUMERICAL INTEGRATION

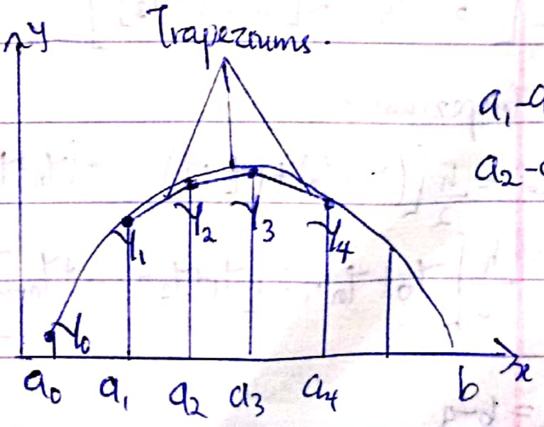
The methods of Numerical integration are:

i) Trapezium Rule.
ii) Mid-Ordinate Rule.
iii) Simpson's Rule.

TRAPEZIUM RULE.



$\int_a^b f(x)\,dx = A$ {Area of the curve}.



$a_1 - a_0 = h$

$a_2 - a_1 = h.$

Area of the first trapezium;

$A = \frac{h}{2}(y_1 + y_0)$

Area of the second Trapezium;

$A = \frac{h}{2}(Y_2 + Y_1)$.

Area of the third Trapezium.

$A = \frac{h}{2}(Y_3 + Y_2)$

Area of the nth Trapezium.

$A_n = \frac{h}{2}(Y_n + Y_{n-1})$.

Total Area $= \frac{h}{2}\left[(Y_1+Y_0)\right] + \frac{h}{2}(Y_2+Y_1) + \frac{h}{2}(Y_3+Y_2)\right]$

$= \frac{h}{2}\left[Y_1+Y_0+Y_2+Y_1+Y_3+Y_2\right]$.

$= \frac{h}{2}\left[Y_0+Y_3+2(Y_1+Y_2)\right]$.

For 4th Trapezium.

Total Area $= \frac{h}{2}\left[(Y_1+Y_0)+(Y_2+Y_1)+(Y_3+Y_2)+(Y_4+Y_3)\right]$

$= \frac{h}{2}\left[Y_0+Y_4+2[Y_1+Y_2+Y_3]\right]$.

For nth Trapezium.

Total Area $= \frac{h}{2}\left[(Y_1+Y_0)+(Y_2+Y_1)\ldots+(Y_n+Y_{n-1})\right]$

$= \frac{h}{2}\left[Y_0+Y_n+2[Y_1+Y_2+\ldots+Y_{n-1}]\right]$

Where $h = \frac{b-a}{n}$

$= \frac{h}{2}\cdot\left[2\left(\frac{Y_0+Y_n}{2}\right) + (Y_1+Y_2+\ldots+Y_{n-1})\right]$

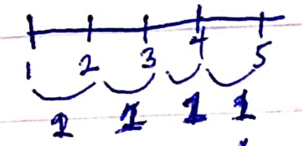
$= h\left[\frac{1}{2}(Y_0+Y_n)\right.$

28-06-18.

Evaluate the Integral $\int_1^5 x^3 dx$ by trapezium ru

Using $n=4$.

Soln.



$h=1$. $h = \frac{b-a}{n}$

By trapezium $A = \frac{h}{2}\left[y_0 + y_4 + 2(y_1+y_2+y_3)\right]$

Where $y = x^3$. $x_0 = 1$, $x_1 = 2$, x

$y_0 = 1$, $y_1 = 8$, $y_2 = 27$, $y_3 = 64$, $y_4 = 125$.

$A = \frac{1}{2}\left[1+125+2(8+27+64)\right]$

$= 162$.

$\int_1^5 x^3 = \frac{x^4}{4}\Big|_1^5 = \frac{5^4}{4} - \frac{1}{4} = 156$] Exact value.

~~Since the~~

$h = \frac{1}{2}$ when n is increased to 8.

$A = \frac{h}{2}(Y_0 + Y_8 + 2(Y_1+Y_2+Y_3+Y_4+Y_5+Y_6+Y_7))$.

$Y_5 = 5^3$

$x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$, $x_5 = 3.5$,

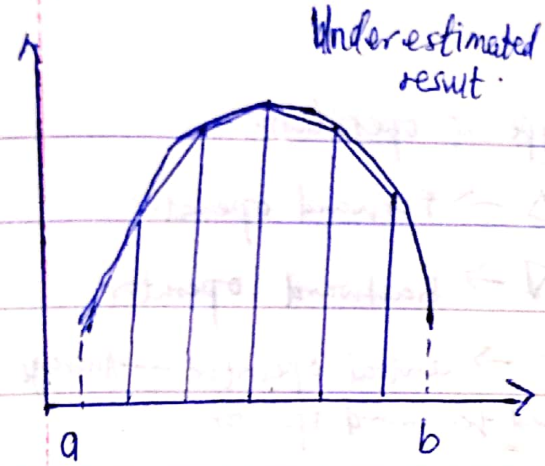
$x_6 = 4$, $x_7 = 4.5$, $x_8 = 5.0$.

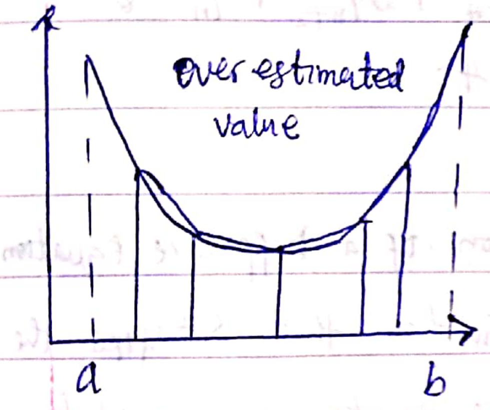
$y_0 = 1$, $y_1 = 3.375$, $y_2 = 8$, $y_3 = \overset{15.625}{\cancel{8.625}}$, $y_4 = 27$

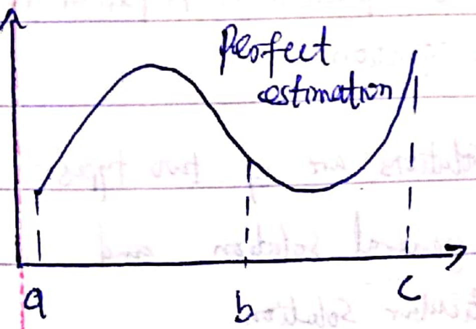
$y_5 = 42.875$ $y_6 = 64$ $y_7 = 91.125$, $y_8 = 125$.

$A = \frac{1}{4}\left[1+125+2(3.375+8+15.625+27+42.875+64+91.125)\right]$.

$A = 157.5$







② MID- POINT ORDINATE.

$A = h(y_1 + y_2 + \dots . y_n)$ . ....

where $x_2 = \frac{x_1 + x_2}{2}$.

③ SIMPSEN'S RULE.

$A = \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + y_5 + ---) + 2(y_2 + y_4 + y_6 + ---)]$.

* Evaluate the integral $\int_1^5 x^3 dx$ using mid point and ~~ordinate~~ Simpson's rule. n=4.

Soln. $h = 1$.

$x_0 = 1, \; x_1 = 2, \; x_2 = 3, \; x_3 = 4, \; x_5 = 5$.

For mid point ordinate,

$A = h[y_1 + y_2 + \dots + y_n]$.

$h = 1, \; x_1 = 1.5, \; x_2 = 2.5, \; x_3 = 3.5, \; x_4 = 4.5$

$y_1 = 3.375, \; y_2 = 15.625, \; y_3 = 42.875, \; y_4 = 91.125$

$A = 1[3.375 + 15.625 + 42.875 + 91.125]$

$A = 153$.

For Simpson's Rule.

$A = \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + y_5 + \dots) + 2(y_2 + y_4 + y_6 + \dots)]$.

$A = \frac{1}{3}[1 + 125 + 4[8 + 64] + 2[27 + 108]]$.

144 108

$= \frac{1}{3}[126 + 4(72) + 2(27)]$.

256 306

$= \frac{1}{3}(126 + 288 + 504)$ 108+126+144

$= ~~\frac{1}{3}(718)~~ = ~~239.33~~$.

$= \frac{1}{3}(468) = 156$ [The Best].

05-07-18 DIFFERENCE EQUATION.

Difference equation is an equation which contains independent variable, dependent variable and a successive difference of the dependent variable.

Example 1:

$$Y_{n+2} + 6Y_{n+1} - 6Y_n = 0$$
$$(E^2 + E + 9)Y_n = 0$$
Both are difference equation.

Where E is an operator.

ORDER OF THE DIFF. EQUATION:

The order of the diff. equation is the difference btwn the largest and the smallest argument that appear in the equation.

DEGREE OF A DIFF. EQUATION:

The degree of a diff. equation is the highest power that appear in the equation.

To get the order ~~of an equation~~ the 1st equation, we use

$$\frac{(n+2) - n}{1} = 2.$$

That shows that the first equation given is of order 2, degree 1.

Order → $\dfrac{\overset{\text{Largest}}{\cancel{\text{Smallest}}}\text{ argument} - \text{Smallest argument}}{\text{No of operator}}$

Type of operation.

Δ → Forward operator

∇ → Backward operator

E → central operator — Average of forward and backward operator.

2) $Y_{n+4} + 3Y_{n+2} + Y_n = 6.$

order 4.

Solution of a Difference Equation.

Any function that satisfies the given difference equation is refered to as solution of the equation.

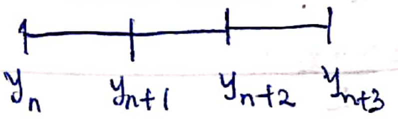
The solutions are of two types:

i) The general solution and

ii) Particular solution

Formation of Difference Equation.

$$\Delta y_n = y_{n+1} - y_n$$



$$\Delta^2 y_n = \Delta(y_{n+1} - y_n) = \Delta(\Delta y_n).$$
$$= \Delta y_{n+1} - \Delta y_n.$$
$$= y_{n+2} - y_{n+1} - (y_{n+1} - y_n)$$
$$= y_{n+2} - y_{n+1} - y_{n+1} + y_n$$

1 6 $\frac{\cdot 6!}{2 \cdot 4!} = \frac{3 \times 5 \times 4!}{2 \cdot 4!}$

$\Delta^2 y_n = y_{n+2} - 2y_{n+1} + y_n$ $\left[\begin{matrix}\text{degree } 1 \\ \text{Order } 2\end{matrix}\right]$.

$\Delta^3 y_n = \Delta(\Delta^2 y_n)$

$= \Delta(y_{n+2} - 2y_{n+1} + y_n)$

$= \Delta y_{n+2} - 2\Delta y_{n+1} + \Delta y_n$.

$= y_{n+3} - y_{n+2} - 2(y_{n+2} - y_{n+1}) + (y_{n+1} - y_n)$

$= y_{n+3} - y_{n+2} - 2y_{n+2} + 2y_{n+1} + y_{n+1} - y_n$

$\Delta^3 y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n$.

$\Delta^6 y_n = y_{n+6} - 6Y_{n+5} + 15Y_{n+4} - 20Y_{n+3}$
$+ 15Y_{n+2} - 6Y_{n+1} + Y_n$.

For backward operator ∇

$\nabla y_n = y_n - y_{n-1}$ [order 1].

For central operator E

$E = \frac{\Delta y_n + \nabla y_n}{2}$

$\therefore E = \frac{y_n - y_{n-1} + y_{n+1} - y_n}{2} = \frac{y_{n+1} - y_{n-1}}{2}$

Central diff. equation for order 1:

Example: Form a diff. equation from

$Y_n = A2^n + B(-2)^n$

$Y_{n+1} = A2^{n+1} + B(-2)^{n+1}$

~~$Y_{n+2} = A2^{n+}$~~

$* Y_{n+1} = (2^n \cdot 2)A + B(-2)^n(-2)$

$Y_{n+1} = A2^n \cdot 2 + B(-2)^n(-2)$

$Y_{n+2} = A2^{n+2} + B(-2)^{n+2}$

$Y_{n+2} = A2^n \cdot 2^2 + B(-2)^n(-2)^2$ *

$Y_{n+2} - A2^n \cdot 2^2 - B(-2)^n(-2)^2$.

$Y_{n+2} = 4Y_n = 2Y_{n+1}$

$Y_{n+1} = 2Y_n$

Linear Difference Equation.

A Difference equation in which $Y_n, Y_{n+2}, \dots$ occur to the first degree only is called a Linear Difference Equation. A Diff. equation

$a_0 Y_{n+K} + a_1 y_{n+K-1} + a_2 y_{n+K-2} + \dots + a_K y_n = r_n$

is said to be Linear non-homogeneous diff. equation but if $r_n = 0$, it is Linear homogeneous equation.

If we write $y_n = M^n$

$(a_0 M^K + a_1 M^{K-1} + a_2 M^{K-2} + \dots + a_K)y_n = 0$

$a_0 M^K + a_1 M^{K-1} + a_2 M^{K+2} + \dots + a_K = 0$ } Auxiliary equation!

If $a_0, a_1, \dots a_K$ are constants then

$M^K + M^{K-1} + M^{K-2} + \dots + 1 = 0$

$M^{n+K} = M^n \cdot M^K = y_n \cdot M^K$.

06-07-18.

$a_0 M^K + a_1 M^{K-1} + a_2 M^{K-2} + \dots + a_K = 0$ is -(ii)

the auxiliary or characteristics equation.

We have three (3) case for the roots of equation II

CASE 1.

If eqn (ii) has real distinct roots, then substitute the solution of (i) is

$$Y_n = C_1(M_1)^n + C_2(M_2)^n + \dots + C_k(M_k)^n$$

where $m_1, M_2 \dots M_k$ are the roots of eqn (ii).

CASE 2: If eqn (2) has equal real roots then the solution of (i) is

$$Y_n = (C_1 + C_2 n)(M_1)^n + C_3(M_3)^n + \dots C_k(M_k)^n .$$

OR.

If three of the ~~equation~~ roots are equal, the solution will be of the form:

$$Y_n = (C_1 + C_2 n + C_3 n^2)(M_1)^n + C_4(M_4)^n + \dots C_k(M_k)^n$$

CASE 3: If eqn (2) has complex roots, then the solution of (1) is

$$Y_n = C_1(\alpha + i\beta)^n + C_2(\alpha - i\beta)^n \oplus$$

$$= \gamma^n(A_1 \cos n\theta + A_2 \sin n\theta) *$$

$$\gamma = \sqrt{\alpha^2+\beta^2}, \quad \theta = \tan^{-1}(\beta/\alpha)$$

$$A_1 = C_1 + C_2, \quad A_2 = i(C_1 - C_2)$$

Example:

Solve the difference equation

$$Y_{n+2} - 2Y_{n+1} - 8Y_n = 0 .$$

Soln

This equation can be written as

$$(E^2 - 2E - 8)Y_n = 0$$

But $Y_n = M^n$.

$$\Rightarrow Y_{n+2} - 2Y_{n+1} - 8Y_n = M^{n+2} - 2M^{n+1} - 8M^n = 0 .$$

$$(M^2 - 2m - 8)M^n = 0 .$$

$M^2 - 2M - 8 = 0$. (Auxillary equation).

$M_1 = -2$ or $M_2 = 4$.

$M_1 = 4, \quad M_2 = -2$.

$$\boxed{Y_n = C_1(4)^n + C_2(-2)^n} \text{ --- (i)}$$

To get the difference equation that generate this solution, we have

$$Y_{n+1} = C_1(4)^{n+1} + C_2(-2)^{n+1}$$

$$= C_1(4)^n \cdot 4 + C_2(-2)^n(-2) \text{ --- (ii)}$$

$$Y_{n+2} = C_1(4)^{n+2} + C_2(-2)^{n+2}$$

$$= C_1(4)^n \cdot 4^2 + C_2(-2)^n \cdot (-2)^2 \text{ --- (iii)}$$

2) $Y_{n+3} + Y_{n+2} - 8Y_{n+1} - 12Y_n = 0$.

Soln.

$$(E^3 + E^2 - 8E - 12)Y_n = 0$$

$$(E^3 + E^2 - 8E - 12)M^n = 0 .$$

$8+4-16$

$-8+4+16-12$

$1+1-8-12$

$\tan\theta \times \frac{1}{0}$

$m^3+m^2-8m-12=0$.

$$\begin{array}{r} m^2-m-6 \\ m+2 \overline{\smash{)}\, m^3+m^2-8m-12} \\ -\underline{m^3+2m^2} \\ -m^2-8m \\ -\underline{-m^2-2m} \\ -6m-12 \\ \underline{-6m-12} \\ \underline{\quad = \quad} \end{array}$$

$(m+2)(m^2-m-6)=0$.

$(m+2)(m+2)(m-3)$

$m_1=-2,\ m_2=-2,\ m_3=3$.

then

$y_n=(c_1+c_2n)(-2)^n+c_3(3)^n$.

3) $y_{n+2}+16y_n=0$.

Soln.

$(E^2+16)y_n=0$.

$(E^2+16)m^n=0$.

$m^2+16=0$.

$m^2=-16$

$m=\pm\sqrt{-16}$

$m_1=4i,\ m_2=-4i$

$\alpha=0,\ \beta=4$.

$r=\sqrt{\alpha^2+\beta^2}=\sqrt{0^2+4^2}=4$.

$\theta=\tan^{-1}\left(\frac{4}{0}\right)$

$y_n=c_1(4i)^n+c_2(-4i)^n$

$=c_1(i)^n4^n+c_2(-i)^n(4)^n$.

$=4^n\left[c_1(i)^n+c_2(-i)^n\right]$.

$\vdots$

$=4^n\left[A\cos(n\pi/2)+B\sin(n\pi/2)\right]$.

NON-HOMOGENEOUS DIFFERENCE EQUATION.

Consider

$(a_0E^k+a_1E^{k-1}+a_2E^{k-2}+\dots+a_k)y_n=r^n$.

The solution here consist of $\underline{y_n^c}$ and $\underline{y_n^p}$.

The Particular Integral.

Case 1: $r(n)=a^n$.

$$P.I=\frac{1}{\Phi(E)}a^n.$$

$\Phi(E)=E^k+E^{k-1}+E^{k-2}+\dots$

$$P.I=\frac{1}{\Phi(a)}a^n.$$

$\Phi(a)=a^k+a^{k-1}+a^{k-2}+\dots$

Provided that $\Phi(a)\neq0$.

If $\Phi(a)=0$, then we have

$$P.I=\frac{1}{(E-a)^k}a^n$$

$$=\frac{n(n-1)(n-2)\dots\dots(n-k+1)}{k!}a^{n-k}$$

$= {}^nC_k a^{n-k}$.

If $k=1$

$P.I = \frac{1}{(E-a)} a^n = {}^nC_1 a^{n-1} = na^{n-1}$.

If $k=2$

$P.I = \frac{1}{(E-a)^2} a^n = {}^nC_2 a^{n-2}$

$= \frac{n(n-1)}{2!} a^{n-2}$.

CASE 2: $r(n) = n^k$.

$P.I = \frac{1}{\Phi(E)} n^k = \frac{1}{\Phi(1+\Delta)} n^k$.

$E = 1+\Delta$.

$P.I = \Phi(1+\Delta)^{-1} n^k$

Example: $y_{n+2} - 4y_{n+1} + 3y_n = 5\cdot4^n$

Soln.

$r(n) = 5\cdot4^n$

$y_{n+2} - 4y_{n+1} + 3y_n = 0$.

$(E^2 - 4E + 3)y_n = 0$.

$(E^2 - 4E + 3)m^n = 0$.

$m^2 - 4m + 3 = 0$.

$m_1 = 1 \quad m_2 = 3$.

$y_n^c = c_1(1)^n + c_2(3)^n$.

$P.I = \frac{1}{\Phi(E)} a^n = \frac{1}{\Phi(E)} 5\cdot4^n$

$= \frac{1}{\Phi(a)} 5\cdot4^n =$

$\Phi(E) = E^2 - 4E + 3$.

$a = 4$.

$P.I = \frac{1}{a^2 - 4a + 3} \cdot 5\cdot4^n$

$= \frac{1}{4^2 - 4(4) + 3} \cdot 5\cdot4^n$

$= 5\cdot\frac{1}{3}\cdot4^n = \frac{5}{3}4^n$

$\therefore y_n = y_n^c + y_n^p$

$y_n = c_1(1)^n + c_2(3)^n + \frac{5}{3}\cdot4^n$.

2) $y_{n+2} - 3y_{n+1} + 2y_n = 6\cdot2^n$.

$r(n) = 6\cdot2^n$

$y_{n+2} - 3y_{n+1} + 2y_n = 0$.

$(E^2 - 3E + 2)y_n = 0$.

$m^2 - 3m + 2 = 0$.

$m_1 = 1, \; m_2 = 2$.

$y_n^c = c_1(1)^n + c_2(2)^n$.

$P.I = \frac{1}{\Phi(E)} 6\cdot2^n = \frac{1}{E^2-3E+2} \times 6\cdot2^n$

$\underline{\Phi}(a) = 0.$

$$= 6 \cdot \frac{1}{E^2 - 3E + 2} \cdot 2^n$$

But $\frac{1}{2^2 - 3(2) + 2} = \frac{1}{0}$,

$\therefore$ The $P \cdot I = \frac{1}{(E-a)} 6 \cdot 2^n.$

$E^2 - 3E + 2 = (E-1)(E-2).$

$$P \cdot I = 6 \cdot \frac{1}{(E-1)(E-2)} 2^n.$$

$E = a = 2.$

$$P \cdot I = 6 \cdot \frac{1}{(E-2)} 2^n$$

where $K = 1.$

$(E-2)^1 = (E-a)^K.$

$$P \cdot I = 6 \cdot \frac{1}{(E-2)^1} 2^n$$

$$= 6 \cdot n(2)^{n-1} = \frac{6n2^n}{2}$$

$P \cdot I = 3n2^n.$

$\therefore Y_n = Y_n^c + Y_n^p$

$* \; Y_n = C_1(1)^n + C_2(2)^n + 3n2^n.$

3) $Y_{n+2} - 6Y_{n+1} + 8Y_n = 3n^2 + 2.$

$(E^2 - 6E + 8) Y_n = 0.$

<u>Soln</u>

$r(n) = 3n^2 + 2.$

$m^2 - 6m + 8 = 0.$

$m_1 = 2, \; m_2 = 4.$

$Y_n^c = C_1(2)^n + C_2(4)^n.$

$$P \cdot I = \frac{1}{\underline{\Phi}(E)} 3n^2 + 2.$$

$n^K = 3n^2 + 2.$

$$P \cdot I = \frac{1}{E^2 - 6E + 8} (3n^2 + 2).$$

$$= \frac{1}{(1+\Delta)^2 - 6(1+\Delta) + 8} (3n^2 + 2)$$

$$= \frac{1}{1 + 2\Delta + \Delta^2 - 6 - 6\Delta + 8} (3n^2 + 2)$$

$$= \frac{1}{\Delta^2 - 4\Delta + 3} (3n^2 + 2)$$

$$= \frac{1}{3 - (4\Delta - \Delta^2)} (3n^2 + 2).$$

$$= [3 - (4\Delta - \Delta^2)]^{-1} \cdot (3n^2 + 2).$$

$$\frac{1}{3}\left[1 + \frac{4\Delta - \Delta^2}{3} + \frac{(4\Delta - \Delta^2)^2}{9} + \cdots\right](3n^2 + 2)$$

Which on Simplification gives

$$= n^2 + \frac{8}{3}n + \frac{44}{9}$$

$\Delta y_n = y_{n+1} - y_n$

$4\Delta(3n^2 + 2) = \Delta(3n^2 + 2)$

$$y_n = C_1(2)^n + C_2(4)^n + n^2 + \frac{8}{3}n + \frac{44}{9}.$$

Exercise.

1) $(E^2 - 2E - 8)y_n = 0$

2) $2y_{n+2} - 5y_{n+1} + 2y_n = 0$.

3) $y_{n+2} - 4y_{n+1} + 3y_n = 4^n$

4) $y_{n+2} - 4y_{n+} = 5\cdot 3^n$

5) $y_{n+2} - 4y_{n+1} + 4y_n = 2^n$.

6) $y_{n+2} - 4y_n = n^2 + n + 1$.

SOLUTION.

1) $m^2 - 2m - 8 = 0$.

$(m \mp 2)(m+4) = 0$.

$m_1 = +2$ and $m_2 = -4$

$y_n = C_1(2)^n + C_2(+4)^n$.

2) $2y_{n+2} - 5y_{n+1} + 2y_n = 0$.

$(2E^2 - 5E + 2)y_n = 0$.

$2m^2 - 5m + 2 = 0$.

$m_1 = -1$ $m_2 = -4$

$y_n = C_1(-1)^n + C_2(-4)^n$

3) $y_{n+2} - 4y_{n+1} + 3y_n = 4^n$

For the homogeneous part,

$y_{n+2} - 4y_{n+1} + 3y_n = 0$.

$(E^2 - 4E + 3)y_n = 0$.

$m^2 - 4m + 3 = 0$.

$m_1 = -1, \quad m_2 = -3$.

$y_n^c = C_1(-1)^n + C_2(-3)^n$ (complementary function).

For Particular integral,

$$P.I = \frac{1}{\Phi(E)} a^n.$$

$$= \frac{1}{E^2 - 4E + 3} \cdot 4^n.$$

$$\frac{1}{\Phi(a)} a^n = \frac{1}{a^2 - 4a + 3} 4^n = \frac{1}{4^2 - 4(4) + 3} \cdot 4^n$$

$$\therefore P.I = \frac{1}{3} 4^n = \frac{4^n}{3}.$$

$$y_n^p = \frac{4^n}{3}.$$

$\Rightarrow y_n = y_n^c + y_n^p$.

$$y_n = C_1(-1)^n + C_2(-3)^n + \frac{4^n}{3}.$$

4) $y_{n+2} - 4y_n = 5\cdot 3^n$.

$(E^2 - 4)y_n = 0$.

$m^2 - 4 = 0$.

$m = \pm 2$.

$\therefore y_n^c = C_1(2)^n + C_2(-2)^n$.

$(1+x)^{-n}$

$(x-1)^{-n}$.

For P.I,

$$P.I = \frac{1}{\Phi(E)} . a^n = \frac{1}{E^2-4} . 5 \cdot 3^n$$

$$= 5 . \frac{1}{a^2-4} 3^n = 5 . \frac{1}{3^2-4} 3^n$$

$$= 5 . \frac{1}{5} . 3^n = 3^n .$$

∴ General solution is

$$Y_n = Y_n^c + Y_n^p = C_1(2)^n + C_2(-2)^n + 3^n .$$

5) $Y_{n+2} - 4Y_{n+1} + 4Y_n = 2^n$.

$(E^2 - 4E + 4)Y_n = 0$.

$m^2 - 4m + 4 = 0$

$m_1 = -2$ twice.

$Y_n^c = (C_1 + C_2 n)(-2)^n$.

$$P.I = \frac{1}{\Phi(E)} . a^n = \frac{1}{E^2-4E+4} . 2^n .$$

$Y_{n+2} - 4Y_n = \Phi(E)$

$$\frac{1}{(E-2)^2} . 2^n$$

For $k=2$.

$$P.I = \frac{n(n-1)}{2} a^{n-2} = \frac{n(n-1)}{2} . 2^{n-2}$$

$$= \frac{n(n-1)}{4} . 2^n .$$

$$Y_n = (C_1 + C_2 n)(-2)^n + \frac{n(n-1)}{4} . 2^n .$$

$(1+\Delta)(1+\Delta)$ $\quad (2\Delta - \Delta^2$

$1 + 2\Delta + \Delta^2$.

6) $Y_{n+2} - 4Y_n = n^2 + n + 1$.

$m^2 - 4 = 0$.

$m = \pm 2$.

$Y_n^c = C_1(-2)^n + C_2(2)^n$.

$$P.I = \frac{1}{\Phi(E)} . n^k$$

$$= \frac{1}{E^2-4} . n^2+n+1$$

But $E = 1 + \Delta$.

$$= \frac{1}{(1+\Delta)^2 - 4} .$$

$$= \frac{1}{1+2\Delta+\Delta^2-4} . n^2+n+1 .$$

$$= \frac{1}{-3+2\Delta+\Delta^2} . (n^2+n+1)$$

$$= \frac{1}{-(3-2\Delta-\Delta^2)} . (n^2+n+1)$$

$$= \frac{-1}{3-2\Delta-\Delta^2} . (n^2+n+1) .$$

$$= -\left[3-2\Delta-\Delta^2\right]^{-1}(n^2+n+1) .$$

$$= \frac{-1}{3}\left[1 - \frac{2\Delta-\Delta^2}{3}\right]^{-1}(n^2+n+1) .$$

$$= \frac{-1}{3}\left[1 + 1\left(\frac{2\Delta-\Delta^2}{3}\right) + \left(\frac{2\Delta-\Delta^2}{3}\right)^2 + \cdots\right](n^2+n+1) .$$

$$= \frac{-1}{3}\left[1 + \frac{2\Delta-\Delta^2}{3} + \frac{(2\Delta-\Delta^2)^2}{9} + \cdot\right](n^2+n+1) .$$

$$= \frac{-1}{3}\left[(n^2+n+1) + \frac{2\Delta-\Delta^2}{3}(n^2+n+1) + \frac{(2\Delta-\Delta^2)^2}{9}(n^2+n+1)\right]$$

$$= -n^2 - n - \frac{10}{3} .$$

$$\Delta(n^2+n+1)$$

$$\Delta n^2+\Delta n$$

$$(n+1)^2-n^2+(n+1)-n$$

$$n^2+2n+1-n^2+1$$

$$2n+2 \quad \Delta 2n$$

$$2(n+1)-2n$$

$$\therefore Y_n^p = -n^2-n-\frac{10}{3}$$

$$\Rightarrow Y_n = C_1(-2)^n + C_2(+2)^n - n^2-n-\frac{10}{3}.$$

$$-\frac{1}{3}\left[1+\frac{2}{3}\Delta-\frac{\Delta^2}{3}+\frac{4\Delta^2-4\Delta^3+\Delta^4}{9}+\cdots\right](n^2+n+1)$$

$$=-\frac{1}{3}\left[1+\frac{2\Delta}{3}-\frac{\Delta^2}{3}+\frac{4}{9}\Delta^2\right](n^2+n+1).$$

$$=-\frac{1}{3}\left(n^2+n+1+\frac{2}{3}\Delta(n^2+n+1)-\frac{\Delta^2}{3}(n^2+n+1)+\frac{4\Delta^2(n^2+n+1)}{9}\right].$$

$$=-\frac{1}{3}\left[n^2+n+1+\frac{2}{3}(2n+2)-\frac{1}{3}(2)+\frac{4}{9}(2)\right]$$

$$=-\frac{1}{3}\left[n^2+n+1+\frac{4}{3}n+\frac{4}{3}-\frac{2}{3}+\frac{8}{9}\right].$$

$$-\frac{n^2}{3}-\frac{n}{3}-\frac{1}{3}-\frac{4n}{9}-\frac{4}{9}+\frac{2}{9}-\frac{8}{27}.$$

$$-\frac{n^2}{3}-\frac{n}{3}-\frac{4n}{9}-\frac{1}{3}-\frac{2}{9}-\frac{8}{27}. \qquad \begin{array}{r}15\\+8\\\hline 23\end{array}$$

$$-\frac{n^2}{3}-\frac{\cancel{n}}{3}-\frac{(3n+4n)}{9}-\frac{9-6-8}{27}$$

$$=-\frac{n^2}{3}-\frac{7n}{9}-\frac{23}{27}.$$

$$Y_n = C_1(-2)^n + C_2(2)^n - \frac{n^2}{3}-\frac{7n}{9}-\frac{23}{27}$$

$$Y_r = {}^rC_0 f_0 + {}^rC_1\Delta f_0 + {}^rC_2\Delta^2 f_0 + {}^rC_3\Delta^3 f_0.$$

$$Y_r = Y(x_0+rh).$$

12-07-18.

NUMERICAL DIFFERENTIATION.

⇒ Newtonian's Formula.

Here, we are concerned with the evaluation of first and higher order derivative of other functions with a given set of values. Whenever values are given to functions and we are interested in evaluating the derivative of such function, we can only use numerical method.

$$Y_r = Y(x_0+rh).$$

Taking the expansion, we have

$$= f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f_0 + \cdots$$

$$\Delta f_0 = f_1 - f_0$$

$$\Delta^2 f_0 = \Delta(\Delta f_0) = \Delta(f_1-f_0) = \Delta f_1 - \Delta f_0$$

$$= f_2-f_1-(f_1-f_0) = f_2-f_1-f_1+f_0$$

$$= f_2-2f_1+f_0.$$

$$\Delta^3 f_0 = \Delta(f_2-2f_1+f_0)$$

$$= \Delta f_2 - 2\Delta f_1 + \Delta f_0 = f_3-f_2-2(f_2-f_1)+f_1-f_0 = f_3-f_2-2f_2+2f_1+f_1-f_0$$

$$= f_3-3f_2+3f_1-f_0.$$

$$X_0 + rh = X$$

$$Y(x_0+rh) = Y(X) .$$

$$\frac{X - X_0}{h} = r$$

$$Y_r' = \frac{d}{dx} y(x) = \frac{d y(x_0+rh)}{dx} \cdot \frac{dx}{dr}$$

$$= \frac{d}{dx} y(x) . h \qquad x = x_0 + rh$$

$$\frac{dx}{dr} = h \Rightarrow \frac{dr}{dx} = \frac{1}{h} . \quad \frac{dy(x)}{dr} \times \frac{dr}{dx}$$

$$\therefore y(x_0+rh) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \cdots$$

$$y_r' = \frac{1}{h}\left[\Delta f_0 + \frac{(2r-1)}{2!}\Delta^2 f_0 + \frac{(3r^2-6r+2)}{3!}\Delta^3 f_0 + \cdots\right]$$

$$12r^2-36r+14$$

$$y_r'' = \frac{1}{h^2}\left[\Delta^2 f_0 + \frac{6r-6}{3!}\Delta^3 f_0 + \frac{(\cancel{12}r^2-3\cancel{6}r+14)}{4!}\Delta^4 f_0 + \cdots\right]$$

$$\frac{d}{dx}\left(\frac{dy(x)}{dx}\right) = \frac{d}{dx}\left(\frac{dy(x)}{dx} \cdot \frac{1}{h}\right) = \frac{dA}{dx} \times \frac{1}{h}$$

$$\frac{dA}{dx} = \frac{dA}{dr} \cdot \frac{dr}{dx} = \frac{dA}{dr} \cdot \frac{1}{h}$$

$$= \frac{dA}{dr} \cdot \frac{1}{h^2}$$

$$r(r-1)(r-2)(r-3) = (r^3-3r^2-2r)(r-3)$$

$$= r^4 - 6r^3 + 7r^2 + 6r$$

$$\frac{d}{dr} = 4r^3 - 18r^2 + 14r + 6 \qquad 4-18+14+6$$

$$y_r' = \frac{1}{h}\left[\Delta + \frac{(2r-1)}{2!}\delta^2 + \frac{(3r^2-6r+2)}{3!}\Delta^3\right] f_0 .$$

$$\frac{3-6+2}{3!}$$

$$\frac{d^2}{dr^2} = 12r^2 - 36r + 14 .$$

y_r' and y_r'' are the first and second order forward Newtonian formula.

Example: Evaluate the first and the second derivative of this table at $x = 1.2$

X	1.0	1.2	1.4	1.6	1.8	2.0	2.2
f(x)	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250
	f_0	f_1	f_2	f_3	f_4	f_5	f_6

Soln.

$h = 0.2$

$x_0 = 1.0$

$$r = \frac{1.2-1}{0.2} = \frac{0.2}{0.2} = 1 .$$

$f_0 = 2.7183$

x	f(x)	Δ	δ^2	Δ^3	Δ^4
1.0	2.7183				
		0.6018			
1.2	3.3201		0.1333		
				0.0294	
		$\boxed{0.7351}$			
1.4	4.0552		$\boxed{0.1627}$		0.0067
				$\boxed{0.0361}$	
		0.8978			
1.6	4.9530		0.1988		0.008
				0.0441	
		1.0966			
1.8	6.0496		0.2429		0.0094
				0.0535	
		1.3395			
2.0	7.3891		0.2964		
		1.6359			
2.2	9.0250				

$(r^4 - 6r^3 + 7r^2 + 6r)(r-4)$ $\frac{31}{\frac{62}{93}}$ $-\frac{1}{2}+2$

$r^5 - 4r^4 - 6r^4 + 24r^3 + 7r^3 - 28r^2 + 6r^2 - 24r$

$5r^4 - 40r^3 + 93r^2 - 44r - 24r$ $\frac{22}{\frac{55}{110}}$

Δ^5 Δ^6

$5 - 40 + 93 - 44 - 24$

$(r^5 - 10r^4 + 31r^3 - 22r^2 - 24r)(r-5)$

0.0013

0.0001 $r^6 - 5r^5 - 10r^5 + 50r^4 + 31r^4 - 155r^3$

$-22r^3 + 110r^2 - 24r^2 + 120r$

0.0014 $6r^5 - 75r^4 + 324r^3 - 531r^2 + 172r + 120$

$6 - 75$

$$y'_r = \frac{1}{0.2}\left[0.7351 + \frac{1}{2}(0.1627) + \left(-\frac{1}{6}\right)0.0361 + \left(\frac{+10}{24}\right)0.008 - \frac{10}{120}(0.0013) + \frac{16}{720}(0.0001)\right]$$

$\times 2.7183$

$$y'_r = \frac{1}{0.2}\left[0.7351 + 0.08135 - 0.00602 + 0.002 - 0.000108 + 2.222\times10^{-6}\right]2.7183$$

$$= \frac{1}{0.2}\left[0.8123242\right]2.7183$$

0.8184522 − 0.006128

$$y'_r = \frac{1}{0.2}(2.20814087286) = 11.0407.$$

y''_r

* $U_n = A + B4^n$.

$U_{n+1} = A + B4^{n+1}$.

$= A + B4^n \cdot 4$

$U_{n+2} = A + B4^{n+2}$.

$A + B4^n \cdot 16$.

$$\begin{vmatrix} \overset{+}{U_n} & 1 & 1 \\ U_{n+1} & 1 & 4 \\ U_{n+2} & 1 & 16 \end{vmatrix} = 0.$$

$U_n[16-4] - U_{n+1}[16-1] + U_{n+2}[4-1] = 0.$

$12U_n - 15U_{n+1} + 3U_{n+2} = 0$.

$U_{n+2} - 5U_{n+1} + 4U_n = 0$.